Asymptotic stability of semigroups associated with linear weak dissipative systems with memory

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Abstract

We study the asymptotic behavior of the solutions of a class of linear dissipative integral differential equations. We show in the abstract setting a necessary and sufficient condition to get an exponential decay of the solution. In the case of the lack of exponential decay, we find the polynomial rate of decay of the solution. Some examples are given.

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1. Introduction

Let us denote by $H$ a Hilbert space. Let $A$, $B$, $C$, $D$ be self-adjoint positive definite operators with the domain $\mathcal{D}(A) \subset \mathcal{D}(D) \subset \mathcal{D}(C) \subset H$ and $\mathcal{D}(A) \subset \mathcal{D}(B) \subset H$ with compact embeddings in $H$. Let $g : [0, \infty) \to [0, \infty)$ be a smooth and summable function. We introduce a class of second-order abstract models

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\[ Du_{tt}(t) + Au(t) - \int_0^\infty g(s) Bu(t - s) \, ds + Cu_t(t) = 0 \quad \text{in } L^2(\mathbb{R}^+; H), \]  

satisfying the initial conditions

\[ u(-t) = u_0(t), \quad t \geq 0, \quad u_t(0) = u_1, \]  

where the initial data \( u_0 \) and \( u_1 \) belong to suitable spaces we will define later. Here the subscript \( \cdot_t \) denotes the time derivative. Following the approach of Dafermos [1], we consider \( \eta = \eta^t(s) \), the relative history of \( u \), defined as

\[ \eta^t(s) = u(t) - u(t - s). \]

Hence, putting

\[ \mu_0 = \int_0^\infty g(s) \, ds, \]  

Eq. (1) turns into the system

\[ Du_{tt}(t) + Au(t) - \mu_0 Bu(t) + \int_0^\infty g(s) B\eta^t(s) \, ds + Cu_t(t) = 0, \]

\[ \eta^t_t = -\eta^t_s + u_t(t). \]  

Accordingly, the initial conditions become

\[ u(0) = u_0, \quad u_t(0) = u_1, \quad \eta^0 = \eta_0, \]  

having set \( u_0 = u_0(0) \) and \( \eta_0(s) = u_0(0) - u_0(s) \), and

\[ \eta^t(0) = \lim_{s \to 0} \eta^t(s) = 0, \quad t \geq 0. \]

The aim of this work is the analysis of the decay properties of system (4)–(6). Our main result characterizes the asymptotic properties of the semigroup in terms of the spectral properties of the operator. To be more precise, we assume that \( D \) is a self-adjoint operator with inverse \( D^{-1} \). Let us denote by

\[ \tilde{A} = D^{-1} A, \quad \tilde{B} = D^{-1} B, \quad \tilde{C} = D^{-1} C, \]

which are also self-adjoint positive definite operators on \( H \). Let us suppose that

\[ \tilde{B} = f(\tilde{A}), \quad \tilde{C} = h(\tilde{A}) \quad \text{with} \quad f(s) = o(s^\beta), \quad h(s) = o(s^\gamma) \quad \text{as} \quad s \to \infty. \]

We show that the semigroup associated with system (4) decays exponentially to zero provided that \( \beta \leq 1 \) and \( 0 \leq \gamma \leq 1 \) (see Section 4). On the other hand, if \( \beta < 1 \) and \( \gamma < 0 \), then the semigroup is not exponentially stable (see Section 5). But in this later case the solution decays polynomially to zero, in an appropriate norm, with rates that can be improved by taking more regular initial data (see Section 6). Finally, in Section 7 we show some applications of our result.

The asymptotic stability of the \( C_0 \)-semigroup associated with the initial value problem (4)–(6) has been studied in [2] in the case \( g \equiv 0 \), where a class of linear dissipative evolution equations is considered. The solution of this class has a polynomial rate of decay and has not the exponential decay as time goes to infinity.

For more details on stability questions for dissipative systems and systems under compact perturbation, we refer the reader to [3–7].
2. Notations and mathematic tools

In the next sections, we use the semigroup approach to show existence and uniqueness of solution to (4)–(6). Then, for \( r \in \mathbb{R} \), we introduce the scale of Hilbert spaces \( \mathcal{D}(\tilde{A}^{r/2}) \), endowed with the usual inner products

\[
(v_1, v_2)_{\mathcal{D}(\tilde{A}^{r/2})} = (\tilde{A}^{r/2}v_1, \tilde{A}^{r/2}v_2).
\]

The embeddings \( \mathcal{D}(\tilde{A}^{r_1/2}) \subset \mathcal{D}(\tilde{A}^{r_2/2}) \) are compact whenever \( r_1 > r_2 \). Analogously, with respect to the other operators \( \tilde{B} \) and \( \tilde{C} \) we can introduce the related scale of Hilbert spaces and the compact embeddings. We suppose also that

\[
\| \tilde{B}^{1/2}u \| \leq k_0 \| \tilde{A}^{1/2}u \|, \quad \forall u \in \mathcal{D}(\tilde{A}^{1/2}),
\]

\[
\| \tilde{C}^{1/2}u \| \leq k_1 \| \tilde{A}^{1/2}u \|, \quad \forall u \in \mathcal{D}(\tilde{A}^{1/2}),
\]

\[
\| \tilde{A}^{1/2}u \|^2 - \mu_0 \| \tilde{A}^{1/2}u \|^2 > k_2 \| \tilde{A}^{1/2}u \|^2, \quad \forall u \in \mathcal{D}(\tilde{A}^{1/2}),
\]

where \( k_i, i = 0, 1, 2 \), are positive constants and the coefficient \( \mu_0 \), introduced in (3), will be assumed positive in (h1). The inner product and the norm on \( H \) are denoted by \( \langle \cdot , \cdot \rangle \) and \( \| \cdot \| \), without subscript.

Concerning the constitutive memory kernel \( g \), recalling (3), we assume the following set of hypotheses:

\[
g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad g(s) \geq 0 \quad \text{and} \quad g'(s) \leq 0, \quad \forall s \in \mathbb{R}^+, \quad \text{(h0)}
\]

\[
\mu_0 > 0. \quad \text{(h1)}
\]

Condition (h1) assures that \( g \) is not identically zero. In fact, the dissipative nature of the system is due also to the presence of the memory kernel.

These hypotheses allow us to introduce the weighted \( L^2 \)-spaces with respect to the measure \( g(s) \, ds \),

\[
\mathcal{M}_j = L^2_g(\mathbb{R}^+, \mathcal{D}(\tilde{B}^{j/2})),
\]

\[
\mathcal{N}_j = L^2_g(\mathbb{R}^+, \mathcal{D}(\tilde{A}^{(1-\beta)/2} \tilde{B}^{j/2})),
\]

endowed with the usual inner products, for some \( \beta < 1 \). Besides (see [8]), let \( T = -\partial_s \) be the linear operator on \( \mathcal{M}_1 \) with domain

\[
\mathcal{D}(T) = \{ \eta \in \mathcal{M}_1: \eta_s \in \mathcal{M}_1, \ \eta(0) = 0 \},
\]

where \( \eta_s \) is the distributional derivative of \( \eta \) with respect to the internal variable \( s \).

To give an accurate formulation of the evolution problem we introduce the product Hilbert spaces

\[
\mathcal{Z} = \mathcal{D}(\tilde{A}^{1/2}) \times H \times \mathcal{M}_1,
\]

endowed with the following inner product (cf. (9))

\[
\langle \varphi, \psi \rangle_{\mathcal{Z}} = \langle \varphi_1, \psi_1 \rangle_{\mathcal{D}(\tilde{A}^{1/2})} - \mu_0 \langle \varphi_1, \psi_1 \rangle_{\mathcal{D}(\tilde{B}^{1/2})} + \langle \varphi_2, \psi_2 \rangle + \langle \varphi_3, \psi_3 \rangle_{\mathcal{M}_1},
\]

where \( \varphi = [\varphi_1, \varphi_2, \varphi_3]^\top, \psi = [\psi_1, \psi_2, \psi_3]^\top \in \mathcal{Z} \).

Moreover, to study the asymptotic behavior of the solution for our evolution problem, we assume that the kernel \( g \) satisfies the following hypothesis

\[
-c_0 g(s) \leq g'(s) \leq -c_1 g(s), \quad \forall s \in \mathbb{R}^+, \quad \text{(h2)}
\]

where \( c_i, i = 0, 1, 2 \), are positive constants.
3. Semigroup of solutions

Following the notations and the assumptions on the operators $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ as introduced in Section 2, system (4) becomes

$$u_{tt}(t) + \tilde{A}u(t) - \mu_0 \tilde{B}u(t) + \int_0^\infty g(s) \tilde{B}\eta'(s) \, ds + \tilde{C}u_t(t) = 0,$$

$$\eta'_t = T\eta'_t + u_t(t).$$

(10)

Setting $v = u_t$ and

$$z(t) = [u(t), v(t), \eta'_t]^\top, \quad z_0 = [u_0, u_1, \eta_0]^\top \in \mathcal{Z},$$

problem (10) can be rewritten as the abstract linear evolution equation in the Hilbert space $\mathcal{Z}$,

$$\begin{cases}
\frac{d}{dt}z(t) = Lz(t), & t > 0, \\
z(0) = z_0,
\end{cases}$$

(11)

where the linear operator $L$ is defined as

$$L \begin{bmatrix}
u \\ u \\ \eta
\end{bmatrix} = \begin{bmatrix}
-\tilde{A}u + \mu_0 \tilde{B}u - \int_0^\infty g(s) \tilde{B}\eta(s) \, ds - \tilde{C}v \\
T\eta + v
\end{bmatrix},$$

with domain

$$D(L) = \left\{ z \in \mathcal{Z}: \begin{array}{c}
\forall u \in D(\tilde{A}), v \in D(\tilde{A}^{1/2}), \\
\int_0^\infty g(s) \tilde{B}\eta(s) \, ds + \tilde{C}v \in H, \ \eta \in D(T)
\end{array} \right\},$$

and the embedding $D(L) \subset \mathcal{Z}$ is compact. First, we will show that $L$ is the infinitesimal generator of a $C_0$-semigroup $S(t) = e^{tL}$ on $\mathcal{Z}$. Then, we will look for the conditions on the operators $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ for which there exists or not exponential stability. In the later case we will find that the solution decays polynomially in appropriate norms.

The energy functional is given by

$$E_1(t) = \frac{1}{2} \| z(t) \|_{\mathcal{Z}}^2 = \frac{1}{2} \left[ \| \tilde{A}^{1/2}u(t) \|^2 - \mu_0 \| \tilde{B}^{1/2}u(t) \|^2 + \| v(t) \|^2 + \| \eta' \|^2_{\mathcal{M}_1} \right].$$

(12)

**Theorem 1.** Assume that the memory kernel $g$ satisfies conditions (h0)–(h1). The linear operator $L$ is the infinitesimal generator of a $C_0$-semigroup $S(t) = e^{tL}$ of contractions on $\mathcal{Z}$.

**Proof.** We will use the Lummer–Phillips theorem (see [9]). First note that the operator $L$ is of dissipative type. In fact, for every $z \in D(L)$, we have

$$\langle Lz, z \rangle_{\mathcal{Z}} = \langle v, u \rangle_{D(\tilde{A}^{1/2})} - \mu_0 \langle v, u \rangle_{D(\tilde{B}^{1/2})}$$

$$+ \left( -\tilde{A}u + \mu_0 \tilde{B}u - \int_0^\infty g(s) \tilde{B}\eta(s) \, ds - \tilde{C}v, v \right) + \langle T\eta + v, \eta \rangle_{\mathcal{M}_1}$$

$$= -\| \tilde{C}^{1/2}v \|^2 + \langle T\eta + v, \eta \rangle_{\mathcal{M}_1}.$$
Integrating by parts, we find
\[
\langle T \eta, \eta \rangle_{\mathcal{M}_1} = \frac{1}{2} \int_0^\infty g'(s) \|	ilde{B}^{1/2} \eta(s)\|^2 \, ds, \quad \forall \eta \in \mathcal{D}(T).
\]
It follows that
\[
\langle Lz, z \rangle_{Z} = -\|	ilde{C}^{1/2} v\|^2 + \frac{1}{2} \int_0^\infty g'(s) \|	ilde{B}^{1/2} \eta(s)\|^2 \, ds \leq 0,
\] (13)
provided \(g'(s) \leq 0\), for any \(s \geq 0\). Next, we show that the operator \(I - L\) is onto. Let \(\hat{z} = [\hat{u}, \hat{v}, \hat{\eta}]^\top \in Z\), and consider the equation
\[
(I - L)z = \hat{z},
\] (14)
or equivalently
\[
u - v = \hat{u},
\] (15)
\[
v + \tilde{A}u - \mu_0 \tilde{B}u + \int_0^\infty g(s) \tilde{B} \eta(s) \, ds + \tilde{C} v = \hat{v},
\] (16)
\[
\eta - T \eta - v = \hat{\eta}.
\] (17)
By (15) we find \(v \in \mathcal{D}(\tilde{A}^{1/2})\). Integration of (17) entails
\[
\eta(s) = (1 - e^{-s}) v + \int_0^s e^{y-s} \hat{\eta}(y) \, dy.
\] (18)
Thus Eq. (16) turns into
\[
u + \left[ \tilde{A} - (\mu_0 - c_g) \tilde{B} + \tilde{C} \right] u
\]
\[
= \hat{v} + \hat{u} + \tilde{C} \hat{u} + c_g \tilde{B} \hat{u} - \int_0^\infty \int_0^s e^{\sigma-s} \tilde{B} \eta(\sigma) \, d\sigma \, ds,
\] (19)
where, from (3) and (h0),
\[
0 < c_g = \int_0^\infty g(s)(1 - e^{-s}) \, ds < \mu_0.
\]
Since the operator \(\tilde{A}\) is self-adjoint and positive definite, and the right-hand side of (19) belongs to \(H\), then there exists only one solution \(u \in \mathcal{D}(\tilde{A}^{1/2})\) of Eq. (19). From (18) we see that \(\eta(s) \in \mathcal{M}_1\), and \(\eta(0) = 0\). Furthermore, from (16) and (17) we get \(\int_0^\infty g(s) \tilde{B} \eta(s) \, ds \in H\) and \(T \eta \in \mathcal{M}_1\). Hence \(z = [u, v, \eta]^\top \in \mathcal{D}(L)\) solves Eq. (14), and our conclusion follows. \(\Box\)
4. Exponential decay

Here we use a result due to Prüss [10], which gives a necessary and sufficient condition to get exponential stability. That is, the semigroup \( T = e^{Lt} \) is exponentially stable if and only if

\[
\{ \lambda \in \mathbb{C}: \text{Re}\, \lambda > 0 \} \subset \rho(L) \quad \text{and} \quad \| (\lambda I - L)^{-1} \| \leq M, \quad \forall \text{Re}\, \lambda \geq 0, \tag{20}
\]

for a fixed \( M > 0 \). To this aim, for \( m \in \mathbb{N} \), let us denote by \( \lambda_m \) the eigenvalues of \( \tilde{A} \) (recall that \( \lambda_m > 0 \) and \( \lambda_m \to \infty \)), and by \( e_m \) the corresponding sequence of eigenvectors, with \( \| e_m \| = 1 \).

To study the asymptotic behavior of the semigroup associated with (11), we assume that the operators \( \tilde{A}, \tilde{B}, \) and \( \tilde{C} \) have the same eigenvectors \( e_m \) and the eigenvalues satisfy

\[
\tilde{A}e_m = \lambda_m e_m, \quad \tilde{B}e_m = \lambda^\beta e_m, \quad \tilde{C}e_m = \lambda^\gamma e_m.
\]

Indeed, there holds

**Theorem 2.** Assume that the memory kernel \( g \) satisfies conditions (h0)–(h2). The semigroup \( S(t) \) on \( \mathcal{Z} \) is exponentially stable, provided that \( \beta \leq 1 \) and \( 0 \leq \gamma \leq 1 \).

**Proof.** Let \( F = [F_1, F_2, F_3]^\top \in \mathcal{Z} \) and consider the following equation

\[
(\lambda I - L)U = F, \quad \lambda \in \mathbb{C}.
\]

This equation reads

\[
\begin{cases}
\lambda u - v = F_1, \\
\lambda v + \bar{A}u - \mu_0 \bar{B}u + \int_0^\infty g(s) \bar{B} \eta(s) \, ds + \bar{C} v = F_2, \\
\lambda \eta - v + \eta_s = F_3.
\end{cases}
\tag{21}
\]

Multiplying the second equation in (21) by \( v \), we find

\[
\langle \lambda, v \rangle + \tilde{A}^{1/2}u, \tilde{A}^{1/2}v \rangle - \mu_0 \tilde{B}^{1/2}u, \tilde{B}^{1/2}v \rangle
\]

\[
+ \int_0^\infty g(s) \langle \tilde{B}^{1/2} \eta(s), \tilde{B}^{1/2} v \rangle \, ds + \| \tilde{C}^{1/2} v \|^2 = \langle F_2, v \rangle.
\]

Multiplying the third equation by \( \int_0^\infty g(s) \tilde{B} \eta(s) \, ds \) we have

\[
\lambda \int_0^\infty g(s) \| \tilde{B}^{1/2} \eta(s) \|^2 \, ds - \mu_0 \int_0^\infty g(s) \| \tilde{B}^{1/2} v, \tilde{B}^{1/2} \eta(s) \| \, ds
\]

\[
+ \int_0^\infty g(s) \| \tilde{B}^{1/2} \eta_s(s), \tilde{B}^{1/2} \eta(s) \| \, ds = \int_0^\infty g(s) \| \tilde{B}^{1/2} F_3, \tilde{B}^{1/2} \eta(s) \| \, ds.
\]

Summing the resulting expressions and substituting \( v \) from the first equation of (21), we obtain

\[
\lambda \| v \|^2 + \bar{\lambda} \| \tilde{A}^{1/2} u \|^2 - \bar{\lambda} \mu_0 \| \tilde{B}^{1/2} u \|^2 + \lambda \| \eta \|_{\mathcal{L}_1}^2 + \| \tilde{C}^{1/2} v \|^2 - \int_0^\infty g'(s) \| \tilde{B}^{1/2} \eta(s) \|^2 \, ds
\]
\[
\int_{0}^{\infty} g(s) \left( \tilde{B}^{1/2} F_3(s), \tilde{B}^{1/2} \eta(s) \right) ds + \langle F_2, v \rangle + \langle \tilde{A}^{1/2} F_1, \tilde{A}^{1/2} u \rangle - \mu_0 \langle \tilde{B}^{1/2} u, \tilde{B}^{1/2} F_1 \rangle. 
\]

From this equality we can estimate
\[
\| \tilde{C}^{1/2} v \|^2 \leq \text{Re} \int_{0}^{\infty} g(s) \left( \tilde{B}^{1/2} F_3(s), \tilde{B}^{1/2} \eta(s) \right) ds + \text{Re} \langle F_2, v \rangle + \text{Re} \langle \tilde{A}^{1/2} F_1, \tilde{A}^{1/2} u \rangle - \mu_0 \text{Re} \langle \tilde{B}^{1/2} u, \tilde{B}^{1/2} F_1 \rangle.
\]

Recalling the hypotheses on \( \tilde{C} \), we have \( \|v\| \leq \lambda_1^{-\gamma} \|\tilde{C}^{1/2} v\| \). Then, by Cauchy–Schwarz inequality the term \( \text{Re} \langle F_2, v \rangle \) can be estimated as
\[
\text{Re} \langle F_2, v \rangle \leq \| F_2 \| \|v\| \leq \| F_2 \| \lambda_1^{-\gamma} \| \tilde{C}^{1/2} v \|.
\]

Applying Young inequality, we find
\[
\frac{1}{2} \| \tilde{C}^{1/2} v \|^2 \leq \varepsilon \left[ \| \tilde{A}^{1/2} u \|^2 + \| \eta \|_{M_1}^2 \right] + \frac{1}{2\varepsilon} \| F_3 \|_{M_1}^2 + \frac{\lambda_1^{-\gamma}}{2} \| F_2 \|^2
\]
\[
+ \frac{1}{2\varepsilon} \| \tilde{A}^{1/2} F_1 \|^2 + \frac{\mu_0^2 k_1^2}{2\varepsilon} \| \tilde{B}^{1/2} F_1 \|^2.
\]

Multiplying the second equation in (21) by \( u \), we find
\[
\lambda \langle v, u \rangle + \| \tilde{A}^{1/2} u \|^2 - \mu_0 \| \tilde{B}^{1/2} u \|^2 + \int_{0}^{\infty} g(s) \left( \tilde{B}^{1/2} \eta(s), \tilde{B}^{1/2} u \right) ds
\]
\[
+ \langle \tilde{C}^{1/2} v, \tilde{C}^{1/2} u \rangle = \langle F_2, u \rangle,
\]
and then
\[
\| \tilde{A}^{1/2} u \|^2 - \mu_0 \| \tilde{B}^{1/2} u \|^2 + \int_{0}^{\infty} g(s) \left( \tilde{B}^{1/2} \eta(s), \tilde{B}^{1/2} u \right) ds
\]
\[
\leq \|v\|^2 + \| \tilde{C}^{1/2} v \| \| \tilde{C}^{1/2} u \| + \langle F_2, u \rangle + | \langle v, F_1 \rangle |.
\]

By (8) it follows
\[
\frac{1}{2} \left( \| \tilde{A}^{1/2} u \|^2 - \mu_0 \| \tilde{B}^{1/2} u \|^2 \right) + \int_{0}^{\infty} g(s) \left( \tilde{B}^{1/2} \eta(s), \tilde{B}^{1/2} u \right) ds
\]
\[
\leq \left( 2\lambda_1^{-\gamma} + \frac{k_1^2}{k_2} \right) \| \tilde{C}^{1/2} v \|^2 + \frac{\lambda_1^{-1}}{k_2} \| F_2 \|^2 + \frac{1}{4} \| F_1 \|^2,
\]
and then
\[
\frac{1}{2} \left( 1 - \varepsilon_1 \right) \left( \| \tilde{A}^{1/2} u \|^2 - \mu_0 \| \tilde{B}^{1/2} u \|^2 \right)
\]
\[
\leq \left( 2\lambda_1^{-\gamma} + \frac{k_1^2}{k_2} \right) \| \tilde{C}^{1/2} v \|^2 + \frac{\mu_0^2 k_0^2}{2\varepsilon_1} \| \eta \|_{M_1}^2 + \frac{1}{4} \| F_1 \|^2 + \frac{\lambda_1^{-1}}{k_2} \| F_2 \|^2.
\]
for some $0 < \varepsilon_1 < 1$. From (23)–(24) we obtain
\[
\frac{1}{2} (1 - \varepsilon_1) (\| \tilde{A}^{1/2} u \|^2 - \mu_0 \| \tilde{B}^{1/2} u \|^2) \leq \left( 2 \lambda_1 - \frac{k_1^2}{k_2^2} \right) \varepsilon_2 \left( \| \tilde{A}^{1/2} u \|^2 + \| \eta \|^2_{\mathcal{M}_1} + \frac{\mu_0^2 k_0^2}{2\varepsilon_1} \| \eta \|^2_{\mathcal{M}_1} + C_{\varepsilon_2} \| F \|_Z^2 \right).
\]
Choosing $\varepsilon_2 = \frac{k_1}{4} (2 \lambda_1 - \frac{k_1^2}{k_2^2})^{-1} (1 - \varepsilon_1)$, it becomes
\[
\frac{1}{4} (1 - \varepsilon_1) (\| \tilde{A}^{1/2} u \|^2 - \mu_0 \| \tilde{B}^{1/2} u \|^2) \leq \left[ \frac{k_2}{4} (1 - \varepsilon_1) + \frac{\mu_0^2 k_0^2}{2\varepsilon_1} \right] \| \eta \|^2_{\mathcal{M}_1} + C_{\varepsilon_2} \| F \|_Z^2. \tag{25}
\]
Adding the real part of (22) to (25), setting $\varepsilon_1 = 1/2$, by hypothesis (h2) and applying Young inequality, we get
\[
\left( \Re \lambda + \frac{\lambda^\gamma}{2} \right) \| v \|^2 + \left( \Re \lambda + \frac{1}{8} \right) (\| \tilde{A}^{1/2} u \|^2 - \mu_0 \| \tilde{B}^{1/2} u \|^2) + (\Re \lambda + c_1) \| \eta \|^2_{\mathcal{M}_1} \leq \varepsilon (\| \tilde{A}^{1/2} u \|^2 + \| \eta \|^2_{\mathcal{M}_1}) + c \| F \|_Z^2,
\]
where $c_1$ is a positive constant depending on $\mu_0, k_0, k_2$. Then, we find
\[
(\Re \lambda + \delta_0) \| z \|^2_Z \leq \varepsilon \| z \|^2_Z + C \| F \|_Z^2,
\]
where $\delta_0 = \min \{ \frac{\lambda^\gamma}{2}, \frac{1}{8}, c_1 \}$. Finally, choosing $\varepsilon < \delta_0$, we have
\[
\| z \|^2_Z \leq \frac{C}{\Re \lambda + \delta_0 - \varepsilon} \| F \|_Z^2.
\]
This implies that
\[
\| (\lambda I - L)^{-1} \| \leq \frac{C}{\Re \lambda + \delta_0 - \varepsilon}. \quad \Box
\]

**Remark 3.** When $\beta = 1$ and $\gamma \leq 1$, we have an exponential stability. This can be proved by the same techniques as in [11].

## 5. Lack of exponential decay

In this section we will prove that the resolvent operator is not uniformly bounded when $\beta < 1$ and $\gamma < 0$. This means that the semigroup $S(t)$ on $Z$ in this case is not exponentially stable. Here to simplify calculations we suppose that the kernel is of the form $g(s) = e^{-\mu s}$, $s \in \mathbb{R}^+$, with $\mu \in \mathbb{R}^+$. By following the notation introduced in Section 4, we assume that the operators $\tilde{A}, \tilde{B}$, and $\tilde{C}$ have the same eigenvectors $e_m$ and the eigenvalues satisfy
\[
\tilde{A} e_m = \lambda_m e_m, \quad \tilde{B} e_m = \lambda_\beta e_m, \quad \tilde{C} e_m = \lambda_\gamma e_m.
\]
Indeed, there holds

**Theorem 4.** Assume that the kernel is of the form $g(s) = e^{-\mu s}$, $s \in \mathbb{R}^+$, with $\mu \in \mathbb{R}^+$. The semigroup $S(t)$ on $Z$ is not exponentially stable, provided that $\beta < 1$ and $\gamma < 0$. 
Proof. To prove that the semigroup $S(t)$ on $Z$ is not exponential stable, we will find a sequence of bounded functions $F_m = [F_{1,m}, F_{2,m}, F_{3,m}]^T \in Z$ for which the corresponding solutions of the resolvent equations is not bounded. This will prove that the resolvent operator is not uniformly bounded. Let us consider the (complex) equation

$$(\omega I - L)U_m = F_m, \quad \omega \in \mathbb{C}.$$ 

To simplify the notation we will omit the subindex $m$. The equation reads

$$\begin{cases}
\omega u - v = F_1, \\
\omega v + Au - \mu_0 \tilde{B}u + \int_0^{\infty} g(s) \tilde{B} \eta(s) \, ds + \tilde{C} v = F_2, \\
\omega \eta - v + \eta_s = F_3.
\end{cases} \quad (26)$$

Set $F_1 = 0$, $F_2 = 0$, and $F_3 = \frac{-\beta + u}{\lambda_m^2} \, e^{-\lambda_m s} e_m$. We look for solutions of the form

$$u = p e_m, \quad v = q e_m, \quad \eta(s) = \varphi(s) e_m,$$

with $p, q \in \mathbb{C}$ and $\varphi \in L^2_g(\mathbb{R}^+)$. Then, system (26) becomes

$$\begin{cases}
\omega p - q = 0, \\
\omega q + \lambda_m p - \mu_0 \lambda_m^\beta p + \int_0^{\infty} g(s) \lambda_m^\beta \varphi(s) \, ds + \lambda_m \gamma^\varphi q = 0, \\
\omega \varphi(s) - q + \varphi_s(s) = \lambda_m^{-\gamma^\varphi} e^{-\lambda_m s}.
\end{cases} \quad (27)$$

From the first two equations of (27) we get

$$\omega^2 p + \lambda_m p - \mu_0 \lambda_m^\beta p + \int_0^{\infty} g(s) \lambda_m^\beta \varphi(s) \, ds + \lambda_m \omega p = 0,$$

and choosing $\omega^2 + \lambda_m^\gamma \omega + \lambda_m = 0$, we obtain $\omega = (-\lambda_m^\gamma \pm \sqrt{\lambda_m^2 - 4 \lambda_m})/2$. Let us denote by

$$\omega_m = \frac{-\lambda_m^\gamma + \sqrt{\lambda_m^2 - 4 \lambda_m}}{2}.$$ 

We have that $\omega_m$ is arbitrarily close to the imaginary axis when $m \to \infty$ provided that $\gamma < 0$. On the other hand, from (28) we have

$$\int_0^{\infty} g(s) \, ds \, p = \int_0^{\infty} g(s) \varphi(s) \, ds. \quad (29)$$

Solving the ordinary differential equation in the third equation of (27), and using the first equation of (27), we get

$$\varphi(s) = e^{-\omega_m s} + p \frac{-\beta + u}{\omega_m - \lambda_m^2} e^{-\lambda_m s} e_m. \quad (30)$$

By initial data, we have $\eta(0) = 0$. Then

$$c = -p \frac{-\beta + u}{\omega_m - \lambda_m^2}.$$
and (30) becomes
\[
\varphi(s) = \left( -p - \frac{-p + \frac{\lambda_m}{2}}{\omega_m - \frac{\lambda_m}{2}} \right) e^{-\omega_m s} + \frac{\lambda_m}{2} e^{-\lambda_m s}. \tag{31}
\]
From (29) and (31), setting
\[
g(s) = e^{-\mu s}, \quad s \in \mathbb{R}^+, \text{ with } \mu \in \mathbb{R}^+,
\]
we get
\[
p = \frac{-\beta + a}{\mu + \lambda_m^2}.
\]
Hence, if \(a > 0\) then we have that
\[
p \approx c \lambda_m^2, \quad \text{as } \lambda_m \to \infty.
\]
Since \(\beta < 1\) there exists \(a > 0\) such that \(\beta + a < 1\). Recalling that \(u = pe_m\), we conclude that
\[
\|u\|_{\mathcal{D}(\tilde{A}^{1/2})} \approx \left( \lambda_m^2 \right)^{1-\beta - a} \to \infty, \quad \text{as } \lambda_m \to \infty,
\]
so that (20) is violated when \(\beta < 1\) and \(\gamma < 0\). □

\section{6. Polynomial decay}

In this section we will study the asymptotic behavior of the solution for problem (10) with initial data (5)–(6) when \(\beta < 1\) and \(\gamma < 0\), provided that the memory kernel \(g\) decays exponentially as time goes to infinity.

Let us introduce the energy functional defined by
\[
\mathcal{E}_2(t) = \frac{1}{2} \| \tilde{A}^{(1-\beta)/2} \varphi(t) \|_2^2.
\]
Recalling (12), from (13) we have \(\frac{d}{dt} \mathcal{E}_1(t) \leq 0, \quad t \in \mathbb{R}^+\), and consequently problem (11) is dissipative. In the previous section the lack of uniform decay of the energy related to our problem is proved. Then, we look for what type of rate of decay we may expect, when \(\beta < 1\) and \(\gamma < 0\), provided that the memory kernel \(g\) decays exponentially as time goes to infinity.

We prove first some lemmas which will play an important role in the proof of the main result (see Theorem 9) of this section. For all the following lemmas, let us assume that initial data
\[
(u_0, u_1, \eta^0) \in \mathcal{D}(\tilde{A}^{1-\beta/2}) \times \mathcal{D}(\tilde{A}^{(1-\beta)/2}) \times \mathcal{N}_1,
\]
and the memory kernel \(g\) satisfies conditions (h0)–(h2).

**Lemma 5.** Let us suppose that \((u, \eta)\) is a solution of (10) with initial data (5)–(6). There holds
\[
\frac{d}{dt} \mathcal{E}_2(t) = -\left\| \tilde{A}^{(1-\beta)/2} \tilde{C}^{1/2} u_t(t) \right\|^2 + \frac{1}{2} \int_0^\infty g'(s) \left\| \tilde{A}^{(1-\beta)/2} \tilde{B}^{1/2} \eta'(s) \right\|^2 ds.
\]

**Proof.** By the same procedure applied in Theorem 1, substituting \(u\) by \(\tilde{A}^{(1-\beta)/2} u\) and \(\eta\) by \(\tilde{A}^{(1-\beta)/2} \eta\), our conclusion follows. □
Let us introduce the functionals
\[
\Psi_1(t) = -\left( u_t(t) + Cu(t), \int_0^\infty g(s)\eta'(s) \, ds \right) + \frac{1}{2} \mu_0 \| \tilde{C}^{1/2}u(t) \|_2^2,
\]
\[
\Psi_2(t) = \langle u_t, u \rangle + \frac{1}{2} \| \tilde{C}^{1/2}u \|_2^2.
\]

**Lemma 6.** Let us suppose that \((u, \eta)\) is a solution of (10) with initial data (5)–(6). There holds
\[
\frac{d\Psi_1}{dt} = -\mu_0 \| u_t \|_2^2 - \left\langle u_t, \int_0^\infty g'(s)\eta(s) \, ds \right\rangle + \left\langle \tilde{A}^{1/2}u, \int_0^\infty g(s)\tilde{A}^{1/2}\eta(s) \, ds \right\rangle - \mu_0 \left\langle \tilde{B}^{1/2}u, \int_0^\infty g(s)\tilde{B}^{1/2}\eta(s) \, ds \right\rangle + \left\| \int_0^\infty g(s)\tilde{B}^{1/2}\eta(s) \, ds \right\|_2^2 - \left\langle \tilde{C}^{1/2}u, \int_0^\infty g'(s)\tilde{C}^{1/2}\eta(s) \, ds \right\rangle.
\]

**Proof.** Multiplying the first equation of (10) by \(\int_0^\infty g(s)\eta(s) \, ds\) we find
\[
\begin{align*}
\left\langle u_t, \int_0^\infty g(s)\eta(s) \, ds \right\rangle + \left\langle A u, \int_0^\infty g(s)\eta(s) \, ds \right\rangle - \mu_0 \left\langle B u, \int_0^\infty g(s)\eta(s) \, ds \right\rangle = J_1 + J_2 = 0.
\end{align*}
\]

By using the second equation of (10) we can write
\[
J_1 = \frac{d}{dt} \left( u_t, \int_0^\infty g(s)\eta(s) \, ds \right) - \left( u_t, \int_0^\infty g(s)\eta_t(s) \, ds \right)
\]
\[
= \frac{d}{dt} \left( u_t, \int_0^\infty g(s)\eta(s) \, ds \right) - \mu_0 \| u_t \|_2^2 + \left( u_t, \int_0^\infty g(s)\eta_t(s) \, ds \right)
\]
\[
= \frac{d}{dt} \left( u_t, \int_0^\infty g(s)\eta(s) \, ds \right) - \mu_0 \| u_t \|_2^2 - \left( u_t, \int_0^\infty g'(s)\eta(s) \, ds \right)
\]
and
\[ J_2 = \frac{d}{dt} \left[ \tilde{C}u, \int_0^{\infty} g(s) \eta(s) \, ds \right] - \left( \tilde{C}u, \int_0^{\infty} g(s) \eta_t(s) \, ds \right) \]

By substituting the values of \( J_1 \) and \( J_2 \) into (32), our conclusion follows. \( \square \)

**Lemma 7.** Let us suppose that \((u, \eta)\) is a solution of (10) with initial data (5)–(6). There holds

\[
\frac{d\Psi_2}{dt} \leq \|u_t\|^2 - \|\tilde{A}^{1/2}u\|^2 + \mu_0 \|\tilde{B}^{1/2}u\|^2 + \|\tilde{C}^{1/2}u\|^2.
\]

**Proof.** Multiplying the first equation of (10) by \( u \) we find

\[
\langle u_{tt}, u \rangle + \langle \tilde{A}u, u \rangle - \mu_0 \langle \tilde{B}u, u \rangle + \left( \int_0^{\infty} g(s) \tilde{B} \eta(s) \, ds, u \right) + \langle \tilde{C}u_t, u \rangle = 0,
\]

and it can be rewritten as

\[
\frac{d}{dt} \langle u_t, u \rangle - \|u_t\|^2 + \|\tilde{A}^{1/2}u\|^2 - \mu_0 \|\tilde{B}^{1/2}u\|^2
\]

\[
+ \left( \int_0^{\infty} g(s) \tilde{B}^{1/2} \eta(s) \, ds, \tilde{B}^{1/2}u \right) + \frac{1}{2} \frac{d}{dt} \|\tilde{C}^{1/2}u\|^2 = 0.
\]

Then, our conclusion follows. \( \square \)

**Lemma 8.** Let us suppose that \((u, \eta)\) is a solution of (10) with initial data (5)–(6). There holds

\[
\frac{d}{dt} \left( \Psi_1 + \frac{\mu_0}{4} \Psi_2 \right) \leq -\frac{\mu_0}{4} \|u_t\|^2 - \frac{k_2}{16} \mu_0 \|\tilde{A}^{1/2}u\|^2 + c \int_0^{\infty} g(s) \|\tilde{A}^{1/2} \eta(s)\|^2 \, ds.
\]

**Proof.** By considering Lemma 6, applying hypothesis (h2) and recalling (7)–(8), we get

\[
\frac{d\Psi_1}{dt} \leq -\mu_0 \|u_t\|^2 + c_0 \|u_t\| \int_0^{\infty} g(s) \eta(s) \, ds
\]

\[
+ \|\tilde{A}^{1/2}u\| \int_0^{\infty} g(s) \tilde{A}^{1/2} \eta(s) \, ds + \mu_0 \|\tilde{B}^{1/2}u\| \int_0^{\infty} g(s) \tilde{B}^{1/2} \eta(s) \, ds
\]

\[
+ \left( \int_0^{\infty} g(s) \tilde{B}^{1/2} \eta(s) \, ds \right)^2 + c_0 \|\tilde{C}^{1/2}u\| \int_0^{\infty} g(s) \tilde{C}^{1/2} \eta(s) \, ds
\]

\[
\leq -\frac{\mu_0}{2} \|u_t\|^2 + c \int_0^{\infty} g(s) \|\tilde{B}^{1/2} \eta(s)\|^2 \, ds
\]
\[ + c \| \tilde{A}^{1/2} u \| \left( \int_0^\infty g(s) \| \tilde{A}^{1/2} \eta(s) \|^2 ds \right)^{1/2}. \]

By using Lemma 7 and the above inequality, our conclusion follows. \( \square \)

Let us now introduce the functional
\[ \mathcal{L}(t) = N E_2(t) + \Psi_1(t) + \frac{\mu_0}{4} \Psi_2(t), \]
with \( N > 0 \) which is chosen suitably large. We can state the main result of this section. The following theorem provides the rate of decay of the energy to model problem (10) in the set of the hypotheses (h0)–(h2) on the kernel \( g \) given in Section 2.

**Theorem 9.** Suppose that the initial data satisfy
\[ (u_0, u_1, \eta^0) \in D(\tilde{A}^{1-\beta/2}) \times D(\tilde{A}^{(1-\beta)/2}) \times \mathcal{N}_1, \]
and the memory kernel \( g \) verifies conditions (h0)–(h2). Then, the solution of Eq. (10), with initial data (5)–(6), decays polynomially to zero. That is, there exists a positive constant \( c \) such that
\[ E_1(t) \leq c E_2(0) \frac{1}{t}. \]

**Proof.** From the inequalities proved in the previous lemmas, we can find
\[ \frac{d}{dt} \mathcal{L} \leq -\gamma E_1 - c \int_0^\infty g(s) \| \tilde{A}^{1/2} \eta(s) \|^2 ds, \]
for some \( \gamma, c > 0 \). Integrating on \( t \), we obtain
\[ \mathcal{L}(t) + \gamma \int_0^t E_1(\tau) d\tau \leq \mathcal{L}(0) \leq c E_2(0), \quad \forall t > 0. \]

In particular, this implies that
\[ \int_0^\infty E_1(\tau) d\tau \leq c E_2(0), \]
for some positive constant \( c \). Finally, we have
\[ \frac{d}{dt} \left[ t E_1(t) \right] = E_1(t) + t \frac{d}{dt} E_1(t) \leq E_1(t), \]
and subsequently,
\[ t E_1(t) \leq \int_0^\infty E_1(\tau) d\tau \leq c E_2(0). \]

This means that
\[ E_1(t) \leq c E_2(0) \frac{1}{t}. \] \( \square \)
Corollary 10. Under the same hypotheses of Theorem 9, there holds
\[ \| S(t)L^{-1+\beta/2} \| \leq \frac{c}{t}, \quad \forall t > 0. \] (33)

Proof. The polynomial decay implies that
\[ \| S(t)z_0 \| \leq \frac{c}{t} \| L^{1-\beta/2}z_0 \| . \]
Setting \( w_0 = L^{1-\beta/2}z_0 \), we can find
\[ \| S(t)L^{-1+\beta/2}w_0 \| \leq \frac{c}{t} \| w_0 \| . \]
and then
\[ \| S(t)L^{-1+\beta/2} \| \leq \frac{c}{t}, \quad \forall t > 0. \]

Finally, we will show that the polynomial rate of decay can be improved if we take more regular initial data. To do this we use the result in [12]. Let \( X \) be a Banach space. We write \( G \in \mathcal{G}(X, M, \omega) \) if the linear operator \( G \) with domain \( D(G) \) generates a strongly continuous semigroup \( (T(t))_{t \geq 0} \) satisfying \( \| T(t) \| \leq M e^{\omega t} \), for \( t \geq 0 \).

Proposition 11. [12, Proposition 3.1]

(a) Assume that \( -A \in \mathcal{G}(X, M, \omega) \). Let \( \varphi \geq 1 \). If \( \| T(t)(d + \varphi)^{-\alpha} \| \leq C t^{-\beta} \) for \( t > 0 \) and some \( \alpha, \beta > 0 \), then \( \| T(t)(d + \varphi)^{-\alpha \varphi} \| \leq C'(\varphi) t^{-\beta \varphi} \) for \( t > 0 \).

(b) Assume that \( -A \in \mathcal{G}(X, M, 0) \) and \( \varphi \) is invertible. Then the following statements are equivalent with a constant \( \alpha > 0 \):
\[ \| T(t)\varphi^{-\alpha} \| \leq C t^{-1}, \quad t > 0, \]
\[ \| T(t)\varphi^{-\alpha \varphi} \| \leq C'(\varphi) t^{-\varphi}, \quad t > 0, \quad \varphi > 0. \]

Proposition 12. Under the same hypotheses of Theorem 9, there holds
\[ \| S(t)L \varphi(-1+\beta/2) \| \leq \frac{c}{t^\varphi}, \quad \forall t > 0, \]
for any \( \varphi \in \mathbb{R}^+ \).

7. Applications

Finally, we list some examples belonging to class system (1) where we assume hypotheses (h0)–(h2) for the kernel \( g \) introduced in Section 2.

Example 13. Let \( H \) be a Hilbert space and \( A \) a self-adjoint positive operator definite with domain \( D(A) \subset H \) with compact embedding in \( H \). Choosing \( D = I, \ B = A^\beta, \ C = A^{-\gamma}, \beta < 1 \) and \( \gamma > 0 \), our problem becomes
\[ u_{tt}(t) + Au(t) - \int_{0}^{\infty} g(s) A^\beta u(t-s) \, ds + A^{-\gamma} u_t(t) = 0 \quad \text{in} \ L^2(\mathbb{R}^+; H), \]
satisfying the initial conditions
\[ u(-t) = u_0(t), \quad \text{with } t \geq 0, \quad u_t(0) = u_1. \]

Since \( D^{-1} = I \), we can identify \( \tilde{A} = A, \tilde{B} = A^\beta \), and \( \tilde{C} = A^{-\gamma} \). By Theorem 4, the related semigroup \( S(t) \) on \( Z = \mathcal{D}(A^{1/2}) \times H \times L^2_{g}(\mathbb{R}^+, \mathcal{D}(A^{\beta/2})) \) is not exponentially stable.

**Example 14.** Let \( H \) be a Hilbert space and \( A \) a self-adjoint positive operator definite with domain \( \mathcal{D}(A) \subset H \) with compact embedding in \( H \). Setting \( D = I, B = A^\beta, C = I, \beta < 1 \), our problem can be written as
\[ u_{tt}(t) + Au(t) - \int_0^\infty g(s)A^\beta u(t-s) \, ds + u_t(t) = 0 \quad \text{in } L^2(\mathbb{R}^+; H), \]
satisfying the initial conditions
\[ u(-t) = u_0(t), \quad \text{with } t \geq 0, \quad u_t(0) = u_1. \]

By Theorem 2, the related semigroup \( S(t) \) on \( Z = \mathcal{D}(A^{1/2}) \times H \times L^2_{g}(\mathbb{R}^+, \mathcal{D}(A^{\beta/2})) \) is exponentially stable.

**Example 15.** Let \( \Omega \) an open bounded subset of \( \mathbb{R}^2 \) with smooth boundary. Let us consider the system
\[ u_{tt}(t) - \Delta u(t) - \int_0^\infty g(s)u(t-s) \, ds + (-\Delta)^{-1}u_t(t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \]
\[ u(t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \]
\[ u(-t) = u_0(t), \quad \text{with } t \geq 0, \quad u_t(0) = u_1 \quad \text{in } \Omega. \]

We define \( A_D : L^2(\Omega) \supset \mathcal{D}(A_D) \rightarrow L^2(\Omega) \) to be \( A_D = -\Delta \), with Dirichlet boundary conditions, viz., \( \mathcal{D}(A_D) = H^2(\Omega) \cap H^1_0(\Omega) \). \( A_D \) is also positive definite, self-adjoint, and by [13] we have \( \mathcal{D}(A_D^{1/2}) = H^1_0(\Omega) \). By Theorem 4, the associated semigroup \( S(t) \) on \( Z = \mathcal{D}(A_D^{1/2}) \times L^2(\Omega) \times L^2_{g}(\mathbb{R}^+, L^2(\Omega)) = H^1_0(\Omega) \times L^2(\Omega) \times L^2_{g}(\mathbb{R}^+, L^2(\Omega)) \) is not exponentially stable.

**Example 16.** Let \( \Omega \) an open bounded subset of \( \mathbb{R}^2 \) with smooth boundary. Let us consider the following evolution problem
\[ u_{tt}(t) - \Delta u(t) - \int_0^\infty g(s)u(t-s) \, ds + u_t(t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \]
\[ u(t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \]
\[ u(-t) = u_0(t), \quad \text{with } t \geq 0, \quad u_t(0) = u_1 \quad \text{in } \Omega. \]

By Theorem 2, the associated semigroup \( S(t) \) on \( Z = \mathcal{D}(A_D^{1/2}) \times L^2(\Omega) \times L^2_{g}(\mathbb{R}^+, L^2(\Omega)) = H^1_0(\Omega) \times L^2(\Omega) \times L^2_{g}(\mathbb{R}^+, L^2(\Omega)) \) is exponentially stable.
Example 17. Let $\Omega$ an open bounded subset of $\mathbb{R}^3$ with smooth boundary. Let us consider the Kirchhoff model whose the evolution system is

$$u_{tt}(t) - \rho \Delta u_{tt}(t) + \Delta^2 u(t) + \int_0^\infty g(s) \Delta u(t - s) \, ds + u_t(t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$u(t) = 0, \quad \Delta u(t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+,$$

$$u(-t) = u_0(t), \quad \text{with } t \geq 0, \quad u_t(0) = u_1 \quad \text{in } \Omega,$$

(34)

where $\rho \in \mathbb{R}^+$. We introduce the positive self-adjoint operator $A : L^2(\Omega) \supset D(A) ightarrow L^2(\Omega)$ defined by

$$Ah = \Delta^2 h, \quad D(A) = \{ h \in H^4(\Omega): h|_{\partial \Omega} = \Delta h|_{\partial \Omega} = 0 \}.$$  

Then (see [13])

$$A^{1/2}h = A_Dh = -\Delta h, \quad D(A^{1/2}) = H^2(\Omega) \cap H^0_0(\Omega).$$

According to these notations, the first equation in (34) can be rewritten abstractly as

$$(I + \rho A^{1/2})u_{tt}(t) + Au(t) - \int_0^\infty g(s) A^{1/2} u(t - s) \, ds + u_t(t) = 0,$$

or

$$u_{tt}(t) + Au(t) - \int_0^\infty g(s) (I + \rho A^{1/2})^{-1} A^{1/2} u(t - s) \, ds + (I + \rho A^{1/2})^{-1} u_t(t) = 0,$$

where we have set

$$\mathbb{A} = (I + \rho A^{1/2})^{-1} A, \quad D(\mathbb{A}) = D(A^{1/2}) = H^2(\Omega) \cap H^0_0(\Omega),$$

which is a positive, self-adjoint operator on the space $D(A^{1/4}_\rho)$, endowed with the following inner product

$$\langle h_1, h_2 \rangle_{D(A^{1/4}_\rho)} = \langle (I + \rho A^{1/2})h_1, h_2 \rangle, \quad h_1, h_2 \in D(A^{1/4}) = H^0_0(\Omega).$$

Note that (cf. [13]) $D(A^{1/4}) = H^1_0(\Omega)$ where, for $h \in H^1_0(\Omega),

$$\| h \|_{D(A^{1/4})} = \| A^{1/4} h \| = \left( \int_{\Omega} |\nabla h|^2 \, d\Omega \right)^{1/2},$$

which is equivalent to the norm $\| h \|_{H^0_0(\Omega)},$ in turn equivalent to the norm

$$\| h \|_{D(\mathbb{A}^{1/4})} = \left( \| h \|^2 + \rho \| A^{1/4} h \|^2 \right)^{1/2} = \left[ \int_{\Omega} (|h|^2 + \rho |\nabla h|^2) \, d\Omega \right]^{1/2}.$$  

Moreover $D(\mathbb{A}^{1/2}) = H^0_0(\Omega)$. By Theorem 4, the semigroup $S(t)$, associated to our system, on $\mathcal{Z} = D(\mathbb{A}^{1/2}) \times L^2(\Omega) \times L^2_g(\mathbb{R}^+, D((I + \rho A^{1/2})^{-1/2} A^{1/4})) = H^0_0(\Omega) \times L^2(\Omega) \times L^2_g(\mathbb{R}^+, L^2(\Omega))$ is not exponentially stable.
Example 18. Let $\Omega$ an open bounded subset of $\mathbb{R}^3$ with smooth boundary. Let us consider the Kirchhoff model whose evolution system is

$$
\begin{align*}
\frac{\partial}{\partial t}u(t) - \rho \Delta u(t) + \Delta^2 u(t) + \int_0^\infty g(s) \Delta u(t-s) \, ds - \Delta u(t) &= 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\
\Delta u(t) = 0, & \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
u(-t) = u_0(t), & \quad \text{with } t \geq 0, \quad u(0) = u_1 \quad \text{in } \Omega,
\end{align*}
$$

where $\rho \in \mathbb{R}^+$. Applying the same procedure as in Example 17 the first equation in our evolution problem can be rewritten as

$$
\begin{align*}
\frac{\partial}{\partial t}u(t) + \Delta u(t) - \int_0^\infty g(s)(I + \rho A^{1/2})^{-1} A^{1/2} u(t-s) \, ds + (I + \rho A^{1/2})^{-1} A^{1/2} u(t) &= 0.
\end{align*}
$$

By Theorem 2, the associated semigroup $S(t)$ on $\mathcal{Z} = D(A^{1/2}) \times L^2(\Omega) \times L^2_g(\mathbb{R}^+, D((I + \rho A^{1/2})^{-1/2} A^{1/4})) = H^1_0(\Omega) \times L^2(\Omega) \times L^2_g(\mathbb{R}^+, L^2(\Omega))$ is exponentially stable.

References