ON THE SPECTRUM DETERMINED GROWTH ASSUMPTION AND THE PERTURBATION OF $C_0$ SEMIGROUPS

GEN-QI XU and DE-XING FENG

The spectrum determined growth property of $C_0$ semigroups in a Banach space is studied. It is shown that if $A$ generates a $C_0$ semigroup in a Banach space $X$, which satisfies the following conditions: 1) for any $\sigma > s(A)$, $\sup\{\|R(\lambda; A)\| \mid \text{Re} \lambda \geq \sigma\} < \infty$; 2) there is a $\sigma_0 > \sigma(A)$ such that $\int_{-\infty}^{\infty} \|R(\sigma_0 + it; A)x\|^2dt < \infty$, $\forall x \in X$, and $\int_{-\infty}^{\infty} \|R(\sigma_0 + it; A^*)f\|^2dt < \infty$, $\forall f \in X^*$, then $\omega(A) = s(A)$. Moreover, it is also shown that if $A = A_0 + B$ is the infinitesimal generator of a $C_0$ semigroup in Hilbert space, where $A_0$ is a discrete operator and $B$ is bounded, then $\omega(A) = s(A)$. Finally the results obtained are applied to wave equation and thermoelastic system.

1. Introduction

Let $A$ be the infinitesimal generator of a $C_0$ semigroup $T(t)$ on a Banach space $X$. As usual, we define the type or growth order of the semigroup $T(t)$ by

$$\omega(A) = \lim_{t \to \infty} \frac{\log\|T(t)\|}{t},$$

and the spectral bound by

$$s(A) = \begin{cases} \sup\{\text{Re} \lambda \mid \lambda \in \sigma(A)\}, & \text{if } \sigma(A) \neq \emptyset, \\ -\infty, & \text{if } \sigma(A) = \emptyset \end{cases}$$

where $\sigma(A)$ denotes the spectrum of $A$. If $X$ is finitely dimensional, it is well known that

$$\omega(A) = s(A).$$

But in the infinite dimensional case, in general, the above equality (1.3) may not hold. From the Hille-Yosida theorem we see that

$$s(A) \leq \omega(A).$$

We notice that $\omega(A)$ describes the growth order of $T(t)$. From the definition of $\omega(A)$, if $\omega > \omega(A)$, then there exists a constant $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$. Therefore the exponential stability of $T(t)$ is equivalent to the condition that $\omega(A) < 0$. Thus the condition (1.3) is important because it gives a practical criterion for exponential stability of $T(t)$, i.e., the exponential stability of $T(t)$ is completely determined by the spectrum of $A$. So the condition (1.3) is usually called the spectrum determined growth assumption.
It is well known that the spectrum determined growth assumption holds for wide classes of semigroups, for example, for analytic or compact semigroups. These classes, however, do not cover applications to hyperbolic PDEs. So it is a very important problem to investigate which conditions will lead to the spectrum determined growth assumption.

In recent years, many authors have made effort and obtained a lot of results on the spectrum determined growth assumption (cf. [1-16]). Among them, there are two important results: Let A generate a $C_0$ semigroup $T(t)$ in Hilbert space $H$, then

$$\omega(A) = \inf \{ \sigma > s(A) \mid \sup_{\tau \in \mathbb{R}} \| R(\sigma + i\tau; A) \| < +\infty \},$$

(cf. [1-4]) and

$$\omega(A) = \inf \{ \sigma > s(A) \mid \int_{-\infty}^{+\infty} \| R(\sigma + i\tau; A)x \|^2 d\tau < \infty, \forall x \in H \}$$

(cf. [5,6]). In general Banach spaces, an important result is that

$$\rho(A) = \sup \{ \sigma(A), \Re \lambda \}$$

(cf. [7,8]), where $\rho(A) = \lim_{t \to +\infty} t^{-1} \log \| \alpha(T(t)) \|$, $\alpha(\cdot)$ denotes the measure of noncompactness of bounded linear operator, and $\sigma_e(A)$ is the essential spectrum of A. A natural question is whether or not the results obtained in Hilbert spaces can be extended to Banach spaces. There many examples which shows that the equality 1.5 is no longer valid in arbitrary Banach space (cf. [9-11]), that is the boundedness of the resolvent of $A$ is not sufficient to characterize the bound of the spectrum of the semigroup generated by $A$. Therefore it would be interesting to know which additional properties of the resolvent of $A$ ensure the spectrum determined growth assumption. In the remaining parts of the paper, we shall discuss this problem. In section 2, we first establish a sufficient condition for the spectrum determined growth property of $C_0$ semigroups in Banach spaces. Then in section 3 we shall investigate the perturbation of $C_0$ semigroups in Hilbert space. We shall prove that if $A = A_0 + B$ is the infinitesimal generator of a $C_0$ semigroup of operator in $H$, where $A_0$ is discrete and $B$ is bounded, then the spectrum determined growth assumption holds. Finally we will illustrate the usefulness of our results through several examples in controlled system.

### 2. Some Results of the Growth Order of $C_0$ Semigroups

**Theorem 2.1** Let $T(t)$ be the $C_0$ semigroup generated by a closed densely defined linear operator $A$ on a complex Banach space $X$. If the following conditions are satisfied:

1) For all $\sigma > s(A)$,

$$\sup \{ \| R(\lambda; A) \| \mid \Re \lambda \geq \sigma \} < \infty,$$

2) There is an $\sigma_0 > \omega(A)$ such that

$$\int_{-\infty}^{+\infty} \| R(\sigma_0 + i\tau; A)x \|^2 d\tau < \infty, \forall x \in X,$$

then the spectrum determined growth assumption holds.
\[ \int_{-\infty}^{+\infty} \| R(\sigma_0 + i\tau; A^*) f \|^2 d\tau < \infty, \forall f \in X^*. \]  

(2.3)

Then \( \omega(A) = s(A) \).

**Proof** We only need to prove that in the case \( s(A) > -\infty \), for any \( \varepsilon > 0 \), there must be \( M_\varepsilon \geq 1 \) such that

\[ \| T(t) \| \leq M_\varepsilon e^{(s(A) + \varepsilon)t}, \quad t \geq 0. \]

Set \( A_1 = A - (s(A) + \varepsilon) \), it is easily seen that the infinitesimal generator of \( T_1(t) = e^{-(s(A) + \varepsilon)t} T(t) \) is \( A_1 \) and \( s(A_1) = -\varepsilon \). Therefore it is enough to prove that the semigroup \( T_1(t) \) is uniformly bounded. For this purpose, we first estimate the norm of the resolvent of \( A_1 \). Since \( R(\lambda; A_1) = R(\lambda + s(A) + \varepsilon; A) \), from hypothesis 1), we know that

\[ \sup \{ \| R(\lambda; A_1) \| : \Re \lambda \geq \sigma > -\varepsilon \} < \infty. \]  

(2.4)

Denote

\[ M = \sup_{\Re \lambda \geq -\varepsilon/2} \| R(\lambda; A_1) \|, \]

then for \( \lambda = \sigma + i\tau \) with \( \Re \lambda \geq -\varepsilon/2 \), we have

\[ \| R(\sigma + i\tau; A_1) x \| \leq (1 + M |\sigma'_0 - \sigma|) \| R(\sigma'_0 + i\tau; A_1) x \|, \]

(2.5)

where \( \sigma'_0 = \sigma_0 + s(A) + \varepsilon \). Thus for any \( \sigma > \omega(A_1), x \in X, f \in X^* \) and \( t > 0 \), we have

\[ \langle T_1(t)x, f \rangle = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{\lambda t} \langle R(\lambda; A_1)x, f \rangle d\lambda \]

\[ = \frac{1}{2\pi i} \lim_{\tau \to +\infty} \left[ \frac{e^{\lambda t}}{t} \langle R(\lambda; A_1)x, f \rangle \big|_{\sigma - i\tau}^{\sigma' + i\tau} + \int_{\sigma - i\tau}^{\sigma' + i\tau} \frac{e^{\lambda t}}{t} \langle R(\lambda; A_1)^2 x, f \rangle d\lambda \right] \]

\[ = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{\lambda t} \langle R(\lambda; A_1)^2 x, f \rangle d\lambda, \]

where we have used the property of Laplace transform: \( \| R(\sigma + i\tau; A_1) x \| \to 0 (|\tau| \to \infty) \).

For any fixed \( t \geq 1 \), the function \( \frac{e^{\lambda t}}{t} \langle R(\lambda; A_1)^2 x, f \rangle \) is analytical in the zone \( \{ \lambda \in \mathbb{C} : -\frac{1}{2} \varepsilon \leq \Re \lambda \leq \sigma \} \). It follows from (2.4) and (2.5) that

\[ \sup_{-\frac{1}{2}\varepsilon \leq \Re \lambda \leq \sigma} | \langle R(\lambda; A_1)x, f \rangle | \leq (1 + |\sigma + \varepsilon| M) \| R(\sigma + i\tau; A_1)x \| \| f \| \to 0 (|\tau| \to \infty). \]

(2.6)

Hence, from the Cauchy integral theory of complex functions, it follows that

\[ \langle T_1(t)x, f \rangle = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{\lambda t}}{t} \langle R(\lambda; A_1)^2 x, f \rangle d\lambda. \]

It follows from (2.2) (2.3) and (2.5) that the above integral is absolutely convergent. Then there is a constant \( \mu(x, f) \) only dependent on \( x \) and \( f \) such that

\[ | \langle T_1(t)x, f \rangle | \leq \frac{\mu(x, f)}{t}, \forall x \in X, f \in X^*, t \geq 1. \]
Therefore by the uniform boundedness principle, there exists a constant $M'$ such that 
$$
\|T_1(t)\| \leq M', \forall t \geq 1.
$$
Setting $M_1 = \max\{M', \sup_{0 \leq t \leq 1} \|T_1(t)\|\}$, we obtain
$$
\|T_1(t)\| \leq M_1, \forall t \geq 0,
$$
and the proof is then finished.

**Remark** In Theorem 2.1 we only consider the case of $s(A) > -\infty$, if $s(A) = -\infty$, in general, the condition 1) of Theorem 2.1 does not hold, but we can define
$$
\omega_R(A) = \inf\{\omega \in \mathbb{R}; \sup_{\Re \lambda \geq \omega} |R(\lambda; A)| < \infty\}
$$
instead of $s(A)$ in Theorem 2.1, and the condition 2) is still satisfied, then $\omega(A) = \omega_R(A)$. The proof is same as above.

**Corollary 2.2** Let $X$ be a Hilbert space. If the condition (2.1) is satisfied, then $\omega(A) = s(A)$.

**Proof** When $A$ is the infinitesimal generator of a $C_0$ semigroup in $H$, for $\sigma > \omega(A)$ and for any $x, y \in H$, we have $\|e^{-\sigma T}x\|, \|e^{-\sigma T^*}y\| \in L^2(\mathbb{R}_+)$. Since $H$ has Fourier type 2, there always hold
$$
\|R(\sigma + i\cdot A)x\| \in L^2(\mathbb{R}) \quad \text{and} \quad \|R(\sigma - i\cdot A^*)y\| \in L^2(\mathbb{R}), \quad \forall x, y \in H.
$$
So the desired assertion follows.

**Remark** In the proof of Corollary 2.2 we have used the notion of Fourier type. Recall that a Banach space $X$ has Fourier type $p$ for some $p$ with $1 \leq p \leq 2$, if the Fourier transform can be extended to a bounded operator from the Lebesgue-Bochner space $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$ ($\frac{1}{p} + \frac{1}{p'} = 1$), i.e., the vector-valued Hausdorff-Young theorem holds for exponent $p$ in $X$. Every Banach space has Fourier type 1 but only a Hilbert space has Fourier type 2 (cf. [18]).

The conclusion of Corollary 2.2 was first given by Gearhart (1978) for contraction semigroup, and obtained independently by Prüss (1984) and Huang (1985) (see [14]).

**Theorem 2.3** Let $T(t)$ be a $C_0$ semigroup on a complex Banach space $X$ and $A$ be its infinitesimal generator. Then the following assertions are equivalent:
1) for any $\sigma > s(A)$,
$$
\sup \{\|R(\lambda; A)\| \mid \Re \lambda \geq \sigma \} < \infty; \tag{2.7}
$$
2) for any $\sigma > s(A)$, and $0 < \varepsilon < \frac{1}{2}(\sigma - s(A))$, the family $\mathcal{F}$ of analytic functions defined by
$$
\mathcal{F} = \{f_{\tau}(z) = \langle R(\sigma + i\tau + z; A)x, f \rangle \mid \forall x \in X, f \in X^*, \|x\| = \|f\| = 1; \tau \in \mathbb{R}, |z| \leq \varepsilon\}, \tag{2.8}
$$
is normal.

**Proof** Assume that the condition 1) holds. For any $\sigma > s(A)$, taking $\varepsilon$ such that $0 < \varepsilon < \frac{1}{2}(\sigma - s(A))$, we have
$$
\sup_{\tau \in \mathbb{R}, |z| \leq \varepsilon} \|R(\sigma + i\tau + z; A)\| \leq M < \infty.
$$
Thus \( \forall x \in X \) and \( f \in X^* \) with \( \|x\| = \|f\| = 1 \),
\[
|f_{\tau}(z)| = |\langle R(\sigma + i\tau + z; A)x, f \rangle| \leq M, \quad \forall \tau \in \mathbb{R}.
\]
According to the Montel's theorem \([19]\), the family \( \mathcal{F} \) is normal.

Now let condition 2) holds. If there is some \( \sigma > s(A) \) such that
\[
\sup_{\tau \in \mathbb{R}} \|R(\sigma + i\tau; A)\| = \infty,
\]
then according to uniform boundedness principle, there exists an \( x \in X \) with \( \|x\| = 1 \) such that
\[
\sup_{\tau \in \mathbb{R}} \|R(\sigma + i\tau; A)x\| = \infty. \tag{2.9}
\]
Thus for each \( \tau \in \mathbb{R} \), there is a functional \( f_{\tau} \in X^* \) with \( \|f_{\tau}\| = 1 \) such that
\[
\langle R(\sigma + i\tau; A)x, f_{\tau} \rangle = \|R(\sigma + i\tau; A)x\|.
\]
Set
\[
\mathcal{F}_1 = \{(R(\sigma + i\tau + z; A)x, f_{\tau}) \mid \tau \in \mathbb{R}, |z| \leq \epsilon\}.
\]
Obviously, \( \mathcal{F}_1 \subset \mathcal{F} \), so \( \mathcal{F}_1 \) is normal. It follows from (2.9) that there exists a sequence \( \{\tau_n \mid n \geq 1\} \subset \mathbb{R} \) such that \( \tau_n \to \infty \) as \( n \to \infty \) and
\[
\lim_{n \to \infty} \|R(\sigma + i\tau_n; A)x\| = \infty.
\]
Denote
\[
f_n(z) = \langle R(\sigma + i\tau_n + z; A)x, f_{\tau_n} \rangle \in \mathcal{F}_1, \quad n \geq 1.
\]
Then there exists a subsequence \( \{f_{n_k}\} \) such that \( f_{n_k} \to \infty \) as \( k \to \infty \). The limit is uniformly on \( |z| \leq \epsilon \). Notice that each analytic function \( f_n \) can be extended to horizontal set \( \mathcal{Z}_\epsilon \):
\[
\mathcal{Z}_\epsilon = \left\{ z \in \mathbb{C} \mid |z| \leq \epsilon \right\} \cup \left\{ z \in \mathbb{C} \mid \text{Re}z \geq 0, \text{Im}z \leq \epsilon \right\}, \tag{2.10}
\]
hence \( f_{n_k}(z) \to \infty \) uniformly on \( \mathcal{Z}_\epsilon \) as \( k \to \infty \). But it is impossible, because for \( \sigma + \text{Re}z > \omega(A) \), there holds
\[
|f_n(z)| = |\langle R(\sigma + i\tau_n + z; A)x, f_{\tau_n} \rangle| \leq \|R(\sigma + i\tau_n + z; A)\| \leq \frac{M}{\text{Re}z + \sigma - \omega(A)}.
\]
Therefore we have
\[
\sup\{\|R(\lambda; A)\| \mid \text{Re}\lambda \geq \sigma\} < \infty.
\]
The proof is then complete. \( \square \)

3. The Perturbation of \( C_0 \) Semigroups in Hilbert Spaces

In this section we always assume that \( H \) is a complex Hilbert space. Denote by \( \mathcal{L}(H) \) the Banach space of all bounded linear operators on \( H \). Let \( A_0 \) be a closed dense defined linear operator in \( H \). Assume that \( A_0 \) satisfies following hypotheses:
(H1) \( \sigma(A_0) \) consists of isolated eigenvalues: \( \sigma(A_0) = \sigma_p(A_0) = \{ \mu_n; \ n \geq 1 \} \), and each eigenvalue of \( A_0 \) is semi-simple, which means that its geometrical multiplicity is equal to the algebraic multiplicity;

(H2) Denote by \( P_n \) the spectral projector corresponding to eigenvalue \( \mu_n \). For each \( x \in H, x = \sum_{n=1}^{\infty} P_n x \), and there exist positive constants \( M \) and \( m \) such that

\[
m \|x\| \leq \left( \sum_{n=1}^{\infty} \|P_n x\|^2 \right)^{\frac{1}{2}} \leq M \|x\|, \forall x \in H.
\]

For example, a normal operator with compact resolvent in \( H \) has these properties.

For \( \lambda \in \mathbb{C}, r > 0 \), denote by \( \Gamma(\lambda, r) \) the disc with radius \( r \) and center at \( \lambda \), i.e.,

\[
\Gamma(\lambda, r) = \{ z \in \mathbb{C}; |z - \lambda| \leq r \}.
\]

**Theorem 3.1** Let \( A_0 \) be a closed dense defined linear operator satisfying the hypotheses (H1) and (H2) and \( B \in \mathcal{L}(H) \). If \( A = VA_0V^{-1} + B \), where \( V, V^{-1} \in \mathcal{L}(H) \), then \( \sigma(A) \subseteq \bigcup_{\mu_n} \Gamma(\mu_n, r) \) with \( r = \frac{M}{m} \|V^{-1}BV\| \).

**Proof** For \( y \in H \), consider the inhomogeneous linear equation

\[
(\lambda - A_0 - V^{-1}BV)f = g,
\]

where \( f = V^{-1}x, g = V^{-1}y \). If \( \lambda \notin \sigma(A_0) \), then

\[
f - R(\lambda; A_0)V^{-1}Bf = R(\lambda; A_0)g.
\]

It is easily seen that if \( \|R(\lambda; A_0)V^{-1}BV\| < 1 \), then

\[
f = [I - R(\lambda; A_0)V^{-1}BV]^{-1}R(\lambda; A_0)g,
\]

which means that \( \lambda \in \rho(A) \).

We now estimate the norm \( \|R(\lambda; A_0)V^{-1}BV\| \). From the hypothesis (H2) we have

\[
f = \sum_{n=1}^{\infty} P_n f, \quad g = \sum_{n=1}^{\infty} P_n g,
\]

and for \( f \in D(A_0) \),

\[
A_0f = \sum_{n=1}^{\infty} \mu_n P_n f,
\]

it follows

\[
R(\lambda; A_0)V^{-1}BVf = \sum_{n=1}^{\infty} \frac{P_n V^{-1}BVf}{\lambda - \mu_n}.
\]

Therefore

\[
m^2 \left\| \sum_{n=1}^{\infty} \frac{P_n V^{-1}BVf}{\lambda - \mu_n} \right\|^2 \leq \sum_{n=1}^{\infty} \left\| \frac{P_n V^{-1}BVf}{\lambda - \mu_n} \right\|^2 \leq M^2 \sum_{n=1}^{\infty} \left\| \frac{P_n V^{-1}BVf}{\lambda - \mu_n} \right\|^2.
\]

Since

\[
\sum_{n=1}^{\infty} \left\| \frac{P_n V^{-1}BVf}{\lambda - \mu_n} \right\|^2 \leq \frac{\sum_{n=1}^{\infty} \|P_n V^{-1}BVf\|^2}{\text{dist}(\lambda, \sigma(A_0))^2},
\]

\[
\sum_{n=1}^{\infty} \|P_n V^{-1}BVf\|^2 \leq \frac{\sum_{n=1}^{\infty} \|P_n V^{-1}BVf\|^2}{\text{dist}(\lambda, \sigma(A_0))^2},
\]

\[
\sum_{n=1}^{\infty} \|P_n V^{-1}BVf\|^2 \leq \frac{\sum_{n=1}^{\infty} \|P_n V^{-1}BVf\|^2}{\text{dist}(\lambda, \sigma(A_0))^2},
\]

\[
\sum_{n=1}^{\infty} \|P_n V^{-1}BVf\|^2 \leq \frac{\sum_{n=1}^{\infty} \|P_n V^{-1}BVf\|^2}{\text{dist}(\lambda, \sigma(A_0))^2},
\]
it yields that
\[ \left\| \sum_{n=1}^{\infty} \frac{P_n V^{-1} B V_f}{\lambda - \mu_n} \right\| \leq \left( \frac{M}{m} \right)^2 \frac{\| V^{-1} B V_f \|}{\text{dist}(\lambda, \sigma(A_0))}. \]
So
\[ \| R(\lambda; A_0) V^{-1} B V_f \| \leq \left( \frac{M}{m} \right) \frac{\| V^{-1} B V_f \|}{\text{dist}(\lambda, \sigma(A_0))}. \]
Thus \( \lambda \in \rho(A) \) when \( \left( \frac{M}{m} \right) \frac{\| V^{-1} B V_f \|}{\text{dist}(\lambda, \sigma(A_0))} < 1 \). Set \( r = \frac{M}{m} \| V^{-1} B \| \), if \( \lambda \notin \bigcup \Gamma(\mu_n, r) \), then \( \text{dist}(\lambda, \sigma(A_0)) > r \), this implies that \( \lambda \in \rho(A) \). Therefore
\[ \sigma(A) \subset \bigcup_{n \geq 1} \Gamma(\mu_n, r). \]

The proof is finished. \[ \square \]

**Remark** Theorem 3.1 is an extension in Hilbert space of Bauer and Fike’s result in [20].

Let \( A_0 \) be an operator satisfying hypotheses \((H1)\) and \((H2)\), and \( F \) be a subset of \( \sigma(A_0) \). Denote \( k = \frac{M}{m} \) and
\[ P_F = \sum_{\mu_n \in F} P_n, \quad Q_F = I - P_F, \]
then \( \| P_F \| \leq k, \| Q_F \| \leq k \), and for any \( \lambda \in \rho(A_0) \), it holds that
\[ \| R(\lambda; A_0 Q_F) \| \leq \frac{k}{\text{dist}(\lambda, \sigma(A_0) \setminus F)}, \]
\[ \| R(\lambda; A_0) Q_F B Q_F \| \leq \frac{k^2 \| B \|}{\text{dist}(\lambda, \sigma(A_0) \setminus F)}. \]

Now set \( A = A_0 + B \) and \( r = 2k^2 \| B \| \). Denote
\[ \Gamma_\lambda = \Gamma(\lambda, r) \cap \sigma(A_0) \]
\[ P_\lambda = P_{\Gamma_\lambda}, \quad Q_\lambda = I - P_\lambda. \]

**Proposition 3.2** Let \( A_0 \) be an operator satisfying hypotheses \((H1)\) and \((H2)\), and \( B \) be bounded. The operators \( P_\lambda, Q_\lambda \) are defined as above. Then the operator \( \lambda I - A_0 Q_\lambda - Q_\lambda B Q_\lambda \) has the bounded inverse on \( Q_\lambda H = \mathcal{R}(Q_\lambda) \), where \( \mathcal{R}(S) \) denote the range of \( S \).

**Proof** It is easily seen that \( A_0 Q_\lambda = Q_\lambda A_0, \sigma(A_0 Q_\lambda) = \sigma(A_0) \setminus \Gamma_\lambda \), This shows that \( \text{dist}(\lambda, \sigma(A_0 Q_\lambda)) \geq r \) and
\[ \| R(\lambda; A_0 Q_\lambda) Q_\lambda B Q_\lambda \| \leq 1/2. \]
Thus the operator \( \lambda I - A_0 Q_\lambda - Q_\lambda B Q_\lambda \) is revertible and
\[ \|(\lambda I - A_0 Q_\lambda - Q_\lambda B Q_\lambda)^{-1}\| \leq \frac{1}{k \| B \|}. \]
For a sufficient small $\varepsilon > 0$, define an operator $H_\lambda(z)$ as follows:

$$H_\lambda(z) = ((\lambda + z)I - A_0Q_\lambda - Q_\lambda BQ_\lambda)^{-1}, \quad |z| \leq \varepsilon.$$ 

We can choose $\varepsilon$ such that $\|H_\lambda(z)\| \leq 6k/r$ for all $|z| \leq \varepsilon$.

**Proposition 3.3** Let $A_0, B, P_\lambda, Q_\lambda$ be the operators as proposition 3.2. Define the mapping $\Delta_\lambda(\cdot) : \Gamma(0,\varepsilon) \to \mathcal{L}(H)$ by

$$\Delta_\lambda(z) = A_0P_\lambda + P_\lambda B P_\lambda + P_\lambda BQ_\lambda H_\lambda(z)Q_\lambda B P_\lambda.$$ 

Then for any $z \in \Gamma(0,\varepsilon)$, the operator $\Delta_\lambda(z)$ is bounded on $P_\lambda H$ and $\Delta_\lambda(z)$ is analytic with respect to $z \in \Gamma(0,\varepsilon)$. Moreover, $\lambda + z \in \rho(A)$ if and only if $(\lambda + z) - \Delta_\lambda(z)$ has bounded inverse on $P_\lambda H$.

**Proof** Obviously, $\Delta_\lambda(z)$ is bounded on $P_\lambda H$ and $\Delta_\lambda(z)$ is analytic with respect to $z \in \Gamma(0,\varepsilon)$. Now we prove the second assertion of Proposition. Consider the resolvent equation

$$(\lambda + z)f - A_0f - Bf = g, \quad \forall g \in H,$$

which is equivalent to

$$(\lambda + z)P_\lambda f - A_0P_\lambda f - P_\lambda BP_\lambda f - P_\lambda BQ_\lambda f = P_\lambda g,$$  \hfill (3.1)

and

$$(\lambda + z)Q_\lambda f - A_0Q_\lambda f - Q_\lambda BQ_\lambda f - Q_\lambda BP_\lambda f = Q_\lambda g.$$  \hfill (3.2)

From (3.2) it follows that

$$Q_\lambda f = H_\lambda(z)Q_\lambda B P_\lambda f + H_\lambda(z)Q_\lambda g.$$  \hfill (3.3)

Hence we have

$$[(\lambda + z) - \Delta_\lambda(z)]P_\lambda f = P_\lambda g + P_\lambda BQ_\lambda H_\lambda(z)Q_\lambda g.$$  \hfill (3.4)

If $(\lambda + z) - \Delta_\lambda(z)$ has bounded inverse on $P_\lambda H$, then

$$P_\lambda f = [(\lambda + z) - \Delta_\lambda(z)]^{-1}[P_\lambda g + P_\lambda BQ_\lambda H_\lambda(z)Q_\lambda g],$$

and

$$f = P_\lambda f + Q_\lambda f = R(\lambda + z; A)g$$

$$= [(\lambda + z) - \Delta_\lambda(z)]^{-1}[P_\lambda g + P_\lambda BQ_\lambda H_\lambda(z)Q_\lambda g] + H_\lambda(z)Q_\lambda g$$

$$+ H_\lambda(z)Q_\lambda B[(\lambda + z) - \Delta_\lambda(z)]^{-1}[P_\lambda g + P_\lambda BQ_\lambda H_\lambda(z)Q_\lambda g].$$

The proof is then complete.

**Theorem 3.4** Let $T(t)$ be a $C_0$ semigroup on complex separable Hilbert space $H$ and $A$ be its infinitesimal generator. Assume $A = A_0 + B$, where $A_0$ is an operator satisfying $(H1)$ and $(H2)$, and $B$ is a bounded linear operator in $H$. Furthermore, assume that the following conditions are satisfied:
1) for a sufficient large constant $M > 0$, the set
\[
\{ \lambda \in \mathbb{C}; |\lambda| \geq M \} \cap \sigma(A_0)
\]
consists of isolated eigenvalues of $A_0$ with finite multiplicity;

2) for any $\sigma > s(A)$,
\[
\sup_{|\tau| \geq M} \| (\sigma + i\tau + z) - \Delta_{\sigma+i\tau}(z) \|^m(\tau) < \infty,
\]
where $m(\tau) = \dim(\mathcal{R}(P_{\sigma+i\tau}))$, and both $P_\lambda$ and $\Delta_\lambda$ are defined as before.

Then $\omega(A) = s(A)$.

**Proof** For any fixed $\sigma > s(A)$, denote $\lambda = \sigma + i\tau$, take $\varepsilon$ such that $0 < \varepsilon < \frac{1}{2}(\sigma - s(A))$, then $\lambda + z \in \rho(A)$ as $|z| \leq \varepsilon$ for all $\tau \in \mathcal{R}$. From Proposition 3.3 it follows that $(\lambda + z) \in \rho(\Delta_\lambda(z))$, and $\| (\lambda + z) - \Delta_\lambda(z) \|$ is uniformly bounded on the disc $|z| \leq \varepsilon$ for all $\tau \in \mathcal{R}$. Particularly, if $|\tau| > M + \| B \|$ and $\dim(P_\lambda H) = m(\tau) < \infty$, then $\forall x \in P_\lambda H$,
\[
[(\lambda + z) - \Delta_\lambda(z)]^{-1} x = \frac{f_\lambda(x, z)}{\det[(\lambda + z) - \Delta_\lambda(z)]},
\]
where $\det[S]$ is the determinant of $S$ and $f_\lambda(x, z)$ is a vector valued analytic function on $|z| \leq \varepsilon$. It follows from [21] that there is a positive constant $\nu$ such that
\[
\| [(\lambda + z) - \Delta_\lambda(z)]^{-1} \| \leq \nu \frac{\| (\lambda + z) - \Delta_\lambda(z) \|^m(\tau)-1}{\| \det[(\lambda + z) - \Delta_\lambda(z)] \|},
\]
and $\nu$ is independent of $\| (\lambda + z) - \Delta_\lambda(z) \|$, depending only on the norm of $H$. Thus it follows that
\[
\| f_\lambda(x, z) \| \leq \nu \| (\lambda + z) - \Delta_\lambda(z) \|^m(\tau)-1 \| x \|,
\]
and
\[
0 < \| \det[(\lambda + z) - \Delta_\lambda(z)] \| \leq \nu \| (\lambda + z) - \Delta_\lambda(z) \|^m(\tau).
\]
By the hypothesis 2) and Montel's theorem (see [19]), it follows that the family of analytic functions
\[
\mathcal{F}_2 = \{ \det[(\lambda + z) - \Delta_\lambda(z)] \mid |\tau| \geq M + \| B \|, |z| \leq \varepsilon \}
\]
is normal. Noticing that for each $\tau \in \mathcal{R}$, $\det[(\lambda + z) - \Delta_\lambda(z)]$ can be extended analytically to the domain $Z_\varepsilon$ defined by (2.10), we have $0 \notin \mathcal{F}_2$. Therefore
\[
\sup_{|\tau| \geq M} \| [(\lambda + z) - \Delta_\lambda(z)]^{-1} \| < \infty.
\]
From Proposition 3.3 and its proof, it follows that
\[
\sup_{\tau \in \mathcal{R}} \| R(\lambda + z; A) \| < \infty,
\]
which means that $\omega(A) = s(A)$ according to Corollary 2.2.
Remark In the proof of Theorem 3.4 we see that it always holds that
\[
\sup_{|\tau| \geq M} \| (\sigma + i\tau + z) - \Delta_{\sigma+i\tau}(z) \| < \infty.
\]
If for sufficiently large $|\tau| \geq M$, $\dim(P_{\lambda} H) = m(\tau) < \text{constant}$, the condition 2) in Theorem 3.4 is satisfied naturally. In practice, many problems such as in [23–27] do satisfy the conditions in Theorem 3.4. Renardy [22] proved that if $A_0$ is a normal operator and for sufficiently large $|\tau| \geq M, m(\tau) \leq K$ hold uniformly, $K$ is a constant, then $\omega(A) = s(A)$. Obviously it is a special case of Theorem 3.4, So our result is a better extension.

Finally, as an extension of Theorem 3.4, we give a more useful result in practice.

**Theorem 3.5** Let $H$ be a separable Hilbert space and $A = A_0 + B$ be the infinitesimal generator of a $C_0$ semigroup in $H$. Assume that $B$ is bounded and $A_0$ satisfies the following conditions:

1) $\sigma(A_0)$ consists of isolated eigenvalues with finite algebraic multiplicity;
2) all eigenvalues of $A_0$ with sufficiently large module are semi-simple;
3) the sequence of the generalized eigenvectors of $A_0$ forms a Riesz basis in $H$;
4) and for any $r > 0$, there exists an integer $k$ such that, for any given $\lambda \in \mathcal{C}$, it holds that
\[
\dim_{\mathbb{C}} \left( \sum_{\mu \in \sigma(A_0) \cap \Gamma(\lambda,r)} P_{\mu} H \right) \leq k,
\]
where $P_{\mu}$ is the spectral projector corresponding to eigenvalue $\mu$, and $\Gamma(\lambda,r)$ defined as before.

Then the spectrum determined growth assumption is valid, i.e., $\omega(A) = s(A)$.

**Proof** Let $\sigma(A) = \{\lambda_n; n \geq 1\}$ with $|\lambda_n| \leq |\lambda_{n+1}|, \forall n \geq 1$, and $P_n$ be the spectral projector corresponding to $\lambda_n$, then from the hypotheses it follows that $\sum_{n=1}^{\infty} P_n = I$ and there is an integer $N$ such that $A_0$ is expressed as
\[
A_0 = \sum_{n=N+1}^{\infty} \lambda_n P_n + \sum_{n=1}^{N} (\lambda_n P_n + D_n),
\]
where $D_n, 1 \leq n \leq N$, is a bounded linear operator with the property that there is an integer $k_n$ such that $D_n^{k_n} = 0$. Denote by $A_{01}$ the scalar part of $A_0$, i.e.,
\[
A_{01} = \sum_{n=1}^{\infty} \lambda_n P_n,
\]
and set $C = A_0 - A_{01}$. Obviously, $C$ is a bounded operator, and $A_{01}$ satisfies all conditions of Theorem 3.4 with $m(\tau) \leq k$. Applying Theorem 3.4 to $A = A_{01} + C + B$, the assertion of Theorem 3.5 follows immediately.

4. Examples

In the present section, we list two examples to explain the application of the results obtained previously.
The first example is concerned in the wave equation with linear damping and tangential force feedback control:

\[
\begin{align*}
W_{tt}(x,t) &= W_{xx}(x,t) - b(x)W_x(x,t) - c(x)W_t(x,t), \quad 0 < x < 1, \ t > 0, \\
W(0,t) &= 0, \ t > 0 \\
W_x(1,t) &= -kW_t(1,t), \ k > 0, \ t > 0,
\end{align*}
\]

(4.1)

where \( b(x), c(x) \) are nonnegative bounded measurable functions.

Set \( V_0^1(0, 1) = \{ f \in H^1(0, 1), f(0) = 0 \} \), where \( H^k(0, 1) \) is the usual Sobolev space of order \( k \). \( \mathcal{H} = V_0^1(0, 1) \times L^2(0, 1) \) is the product Hilbert space with the norm

\[
\| (f, g) \|^2 = \int_0^1 \left( |f_x(x)|^2 + |g(x)|^2 \right) dx, \quad \text{for} \quad f \in V_0^1, g \in L^2(0, 1).
\]

Define operators \( A_k \) and \( B \) in \( \mathcal{H} \) by

\[
A_k(f, g) = (g, f_{xx}),
\]

\[
\mathcal{D}(A_k) = \{(f, g) \in \mathcal{H}; f \in H^2 \cap V_0^1, g \in V_0^1, f_x(1) = -kg(1)\},
\]

and

\[
B(f, g) = (0, -b(x)f_x(x) - c(x)g(x)), \quad \forall (f, g) \in \mathcal{H}.
\]

Obviously \( B \) is bounded. Set \( A = A_k + B \). Then the equation (4.1) can be written as the abstract evolution equation in \( \mathcal{H} \):

\[
\frac{d}{dt} U(t) = AU(t) \quad \text{with} \quad U(t) = (W(x,t), W_t(x,t)), \quad \text{for} \quad t > 0.
\]

It is easily seen that \( A \) is the infinitesimal generator of a \( C_0 \) semigroup.

It is shown in [28] that for all \( k > 1 \), the operator \( A_k \) has compact resolvent and the spectrum of \( A_k \) is

\[
\sigma(A_k) = \{-\sigma \pm i\omega_n; \sigma = \frac{1}{2} \ln \frac{1+k}{1-k}, \omega_n = \frac{(2n+1)\pi}{2}, \ n \geq 0\},
\]

and every eigenvalue of \( A_k \) is algebraically simple. The sequence of eigenvectors of \( A_k \) is a Riesz basis in \( \mathcal{H} \).

Thus it follows that the operator \( A_k \) satisfies the all conditions of theorem 3.4, so we have \( \omega(A) = s(A) \).

Another example is concerned in the linear thermoelastic system with damping:

\[
\begin{align*}
\nu u_{tt}(x,t) + \nu \theta_x(x,t) &= u_{xx}(x,t) - b(x)u_x(x,t) - c(x)u_t(x,t), \quad 0 < x < 1, \ t > 0 \\
\theta_t(x,t) + \nu u_{xx}(x,t) - k\theta_{xx}(x,t) &= 0, \ 0 < x < 1, \ t > 0, \\
u(0,t) &= u(1,t) = 0, \ t > 0, \\
\theta(0,t) &= \theta(1,t) = 0, \ t > 0,
\end{align*}
\]

(4.2)

where \( \nu > 0, \ k > 0 \) are constants, \( \nu \) usually much less than 1, and \( b(x), c(x) \) are nonnegative bounded measurable functions.
Set \( H_0^1(0, 1) = \{ f \in H^1(0, 1), f(0) = 0, f(1) = 0 \} \), \( \mathcal{H} = H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \) is the product Hilbert space equipped with the norm

\[
\|(f, g, h)\|_1^2 = \int_0^1 \left( |f_x(x)|^2 + |g(x)|^2 + |h(x)|^2 \right) dx, \quad (f, g, h) \in \mathcal{H}.
\]

Define operators \( A_0 \) and \( B \) in \( \mathcal{H} \) by

\[
A_0(f, g, h) = (g, f_{xx} - \nu h_x, -\nu g_x + k h_{xx})
\]

\[
D(A_0) = \left\{ (f, g, h) \in \mathcal{H}; f \in H^2 \cap H_0^1, g \in H_0^1, h \in H^2 \cap H_0^1, \right\},
\]

and

\[
B(f, g, h) = (0, -b(x)f_x(x) - c(x)g(x), 0), \quad \forall (f, g, h) \in \mathcal{H}.
\]

Then \( B \) is bounded, and \( A = A_0 + B \) is the infinitesimal generator of a \( C_0 \) semigroup.

It is pointed out in [29] that the set of generalized eigenfunctions of \( A_0 \) forms a Riesz basis in \( \mathcal{H} \), in addition, all eigenvalues with sufficiently large module are algebraically simple. Furthermore, \( A_0^{-1} \) exists, and the eigenvalues of \( A_0 \) consist of a real sequence \( \{\sigma_n\} \) and a sequence of conjugate pairs \( \{\lambda_n, \bar{\lambda}_n\} \) with the following asymptotic properties

\[
\sigma_n = -k(n\pi)^2 + \nu^2 + o\left(\frac{1}{n}\right),
\]

\[
\lambda_n = -\frac{\nu^2}{2k} + in\pi + o\left(\frac{1}{n}\right),
\]

for positive integer \( n \) large enough.

It is known from [29] that the operator \( A_0 \) satisfies all conditions of Theorem 3.5, so we can assert that \( \omega(A) = s(A) \).

Acknowledgment This research was supported by the National Key Project of China and the Natural Science Foundation of Shanxi.

References


Institute of Systems Science
Academy of Mathematics and Systems Control
Chinese Academy of Sciences, Beijing 100080
Email: dxfeng@iss03.iss.ac.cn
Fax: (8610)62587343

AMS Subject Classification: 47D06, 47D03

Submitted: November 17, 1999
Revised: May 9, 2000