Geodesics in first-passage percolation

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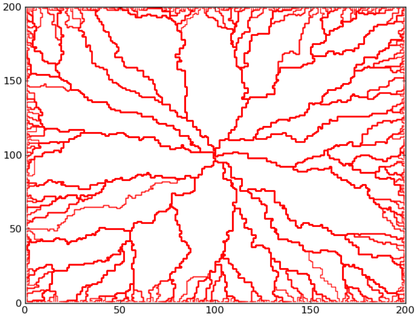
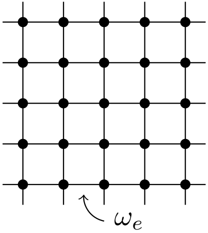


Based on joint work with Christopher Hoffman (and Jack Hanson).

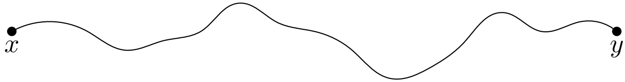
A model for spatial growth



In **first-passage percolation** the edges of $\mathbb{Z}^2$ are assigned i.i.d. weights $\omega_e \geq 0$ from a continuous distribution with finite mean. A random metric:

$$T(x, y) := \inf\Big\{\sum_{e\in\pi} \omega_e : \pi \text{ is a path from } x \text{ to } y\Big\}.$$

Goal: Understand the asymptotics of distances, balls and geodesics.



Subadditive ergodic theory

Key property: $T(x, y) \leq T(x, z) + T(z, y)$ for all $x, y, z \in \mathbb{Z}^2$.

Hammersley-Welsh (1965): $\exists \mu(z) := \lim_{n\to\infty} \frac{1}{n} T(0, nz)$ in probability.

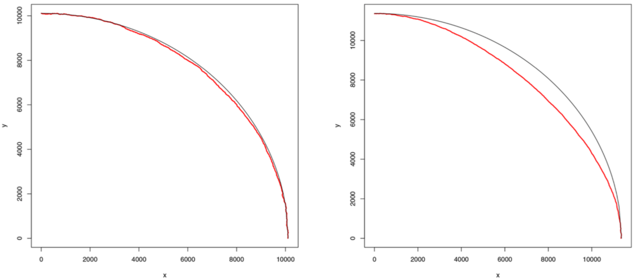
Kingman (1968): The limit exists almost surely.

The shape theorem

Richardson (1973), Cox-Durrett (1981): There exists a compact and convex set Ball $\subset \mathbb{R}^2$ such that, almost surely, for all large t

$$(1-\varepsilon)\text{Ball} \subset \frac{1}{t}\{z \in \mathbb{Z}^2 : T(0,z) \le t\} \subset (1+\varepsilon)\text{Ball}.$$

First Passage Percolation on $\mathbb{Z}^2$: A Simulation Study Alm-Deijfen (2015)



(a) Exp(1) **(b)** 0.5+Exp(1)

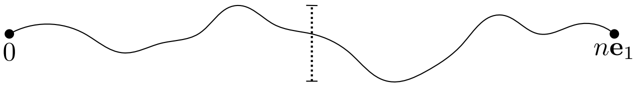
KPZ universality

Kardar-Parisi-Zhang (1986): Predictions due to physicists suggest that

$T(0, n\mathbf{e}_1)$ fluctuates around its mean by order n^χ

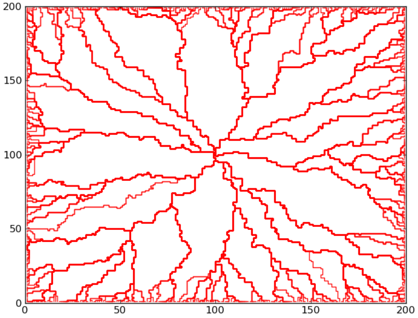
$\mathrm{Geo}(0, n\mathbf{e}_1)$ fluctuates vertically by order n^ξ

where the exponents should equal $\chi = 1/3$ and $\xi = 2/3$, so $\chi = 2\xi - 1$.



Geodesics in first-passage percolation

The **geodesic** between x and y is the path whose weight-sum equals $T(x, y)$. Consider the geodesics from the origin to sites at distance n. We want to describe the geometry of this object when n is large.



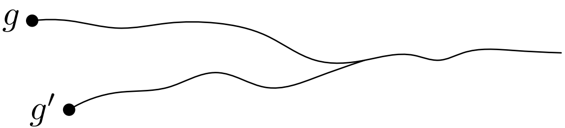
(Simulation for exponential weights, from math*overflow*.)

Geodesics in first-passage percolation

An infinite path is an **infinite geodesic** if every finite segment is a geodesic. A geodesic $g = (v_1, v_2, \ldots)$ has **asymptotic direction** θ if

$$\lim_{k \to \infty} \frac{v_k}{|v_k|} = \theta.$$

Two infinite geodesics g and g' **coalesce** if $g \Delta g'$ is finite.



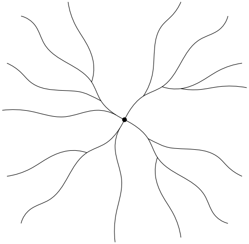
Newman's conjectures

Conditional work of **Newman (1995)** led to the following conjectures:

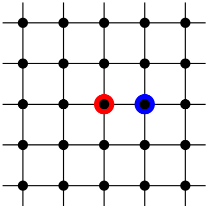
(I) With probability one, *every* infinite geodesic has a direction.

(II) For *every* θ there is an a.s. *unique* geodesic in $\mathscr{T}_0$ with direction θ.

(III) For every θ any two geodesics with direction θ *coalesce* a.s.



A model for competing growth



In the **two-type Richardson model** we initially color $(0,0)$ **red** and $(1,0)$ **blue**. As time evolves, uncolored sites of $\mathbb{Z}^2$ turn

$$\textbf{red} \text{ at rate } 1 \cdot \#\{\textbf{red} \text{ neighbors}\}$$

$$\textbf{blue} \text{ at rate } \lambda \cdot \#\{\textbf{blue} \text{ neighbors}\}$$

A colored site keeps its color forever.

Central question: For which values of $\lambda \geq 1$ is it possible for both **red** and **blue** to conquer infinitely many sites?

Coexistence and existence of multiple geodesics

Let $\mathscr{T}_0$ denote the set of infinite geodesics starting at the origin.

Häggström-Pemantle (1998):

(i) When $\lambda = 1$, coexistence occurs with positive probability.

(ii) For exponential weights, $\mathbb{P}(|\mathscr{T}_0| \geq 2) > 0.064$.

Hoffman (2008): $\mathbb{P}(|\mathscr{T}_0| \geq 4) > 0$.

Damron-Hanson (2014): $\mathbb{P}(|\mathscr{T}_0| \geq 4) = 1$.

Busemann functions

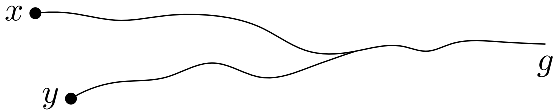
For a geodesic $g = (v_1, v_2, \ldots)$ we define its **Busemann function** as

$$B_g(x, y) := \lim_{k\to\infty} \big[T(x, v_k) - T(y, v_k)\big].$$

The limit exists for all g and satisfies

$B_g(0, y) = T(0, y)$ for all $y \in g$.

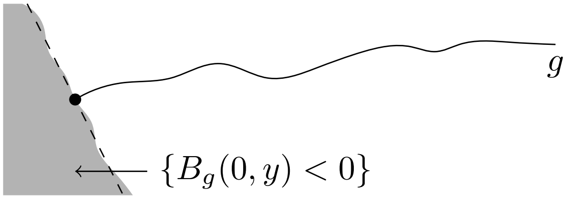
$B_g(0, y) < 0$ iff y further 'from infinity' than the origin along g.



Busemann functions

A linear functional $\rho : \mathbb{R}^2 \to \mathbb{R}$ is called **supporting** if the line $\{x \in \mathbb{R}^2 : \rho(x) = 1\}$ is a supporting line to Ball. Given a supporting functional ρ and a geodesic g we say that the Busemann function of g is **asymptotically linear** to ρ if

$$\limsup_{|y| \to \infty} \frac{1}{|y|} \left| B_g(0, y) - \rho(y) \right| = 0.$$



Asymptotic directions

From ρ we can read out the **direction** of $g = (v_1, v_2, \ldots)$.

$$\rho(v_k/|v_k|) \approx \frac{1}{|v_k|} B_g(0, v_k) = \frac{1}{|v_k|} T(0, v_k) \approx \mu(v_k/|v_k|).$$

So any limit point x of $(\frac{v_k}{|v_k|})_{k\geq 1}$ must satisfy $\rho(x) = \mu(x)$.

Damron-Hanson (2014): For every **tangent** functional ρ there exists a geodesic in $\mathscr{T}_0$ with Busemann function linear to ρ.

Newman's conjectures

Conditional work of **Newman (1995)** led to the following conjectures:

(I) With probability one, *every* infinite geodesic has a direction.

(II) For *every* θ there is an a.s. *unique* geodesic in $\mathscr{T}_0$ with direction θ.

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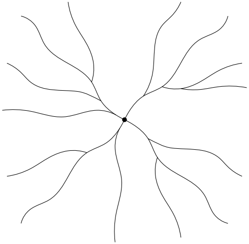
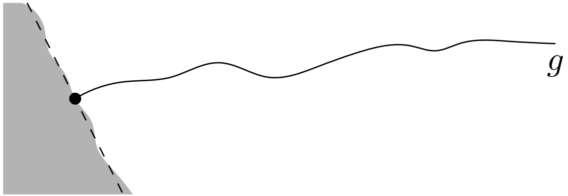


Versions of Newman's conjectures

(I) With probability one, *every* infinite geodesic has a direction.

Theorem I: (A.-Hoffman) With probability one, every infinite geodesic has a linear Busemann function.



Versions of Newman's conjectures

(II) For every direction θ there is an a.s. *unique* geodesic with direction θ.

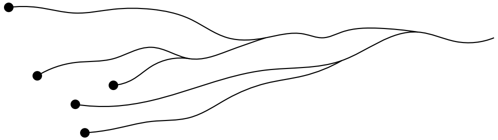
Theorem II: (A.-Hoffman) There is a deterministic set $\mathscr{C}$ such that, a.s., the set of functionals ρ for which there exists a geodesic in $\mathscr{T}_0$ with Busemann function linear to ρ equals $\mathscr{C}$. Moreover, for every $\rho \in \mathscr{C}$

$$\mathbb{P}\big(\exists \text{ two geodesics in } \mathscr{T}_0 \text{ with Busemann function linear to } \rho\big) = 0.$$

Versions of Newman's conjectures

(III) For every direction θ any two geodesics with direction θ *coalesce* a.s.

Theorem III: (A.-Hoffman) For every $\rho \in \mathscr{C}$, any two geodesics with Busemann function linear to ρ coalesce a.s.



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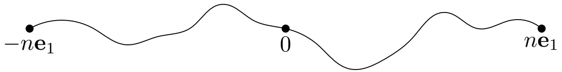
Application I: The midpoint problem

Benjamini-Kalai-Schramm (2003): Does the geodesic between $(-n, 0)$ and $(n, 0)$ visit the midpoint?

Theorem: (A.-Hoffman) For diverging sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$

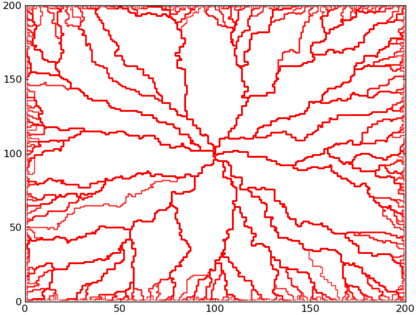
$$\mathbb{P}\big(0 \in \mathrm{Geo}(u_n, v_n)\big) \to 0 \quad \text{as } n \to \infty.$$



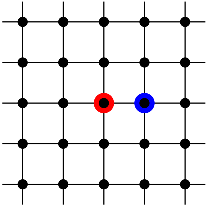
Application II: The highways and byways problem

Hammersley-Welsh (1965): What fraction of points at distance n from the origin lie on geodesics in $\mathscr{T}_0$?

Theorem: (A.-Hanson-Hoffman) The expected fraction tends to zero.



Application III: Existence and coexistence



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$$\textbf{red} \text{ at rate } 1 \cdot \#\{\textbf{red} \text{ neighbors}\}$$

$$\textbf{blue} \text{ at rate } \lambda \cdot \#\{\textbf{blue} \text{ neighbors}\}$$

Equivalent to FPP with exponential weights.

Corollary: (A.) For $\lambda = 1$ and $k \geq 1$ (including $k = \infty$) we have

$$\mathbb{P}(|\mathscr{T}_0| \geq k) > 0 \quad \Leftrightarrow \quad \exists x_1, x_2, \ldots, x_k \text{ s.t. } \mathbb{P}(\mathrm{Coex}(x_1, x_2, \ldots, x_k)) > 0.$$

Application III: Proof

$\Leftarrow$: On the event $\text{Coex}(x_1, x_2, \ldots, x_k)$, there are k disjoint infinite geodesics. Since disjoint they correspond to different functionals ρ. Since the set of functionals is constant, we have $|\mathscr{T}_0| = |\mathscr{C}| \geq k$.

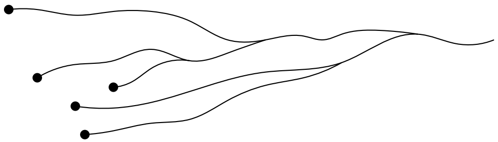
$\Rightarrow$: Suppose $|\mathscr{T}_0| \geq k$. Pick distinct functionals $\rho_1, \rho_2, \ldots, \rho_k$ in $\mathscr{C}$. Position k points $x_1, x_2, \ldots, x_k$ at distance n from the origin in directions given by the gradients of $\rho_1, \rho_2, \ldots, \rho_k$. Since Busemann functions are linear, for large n we have for every $i = 1, 2, \ldots, k$

$$B_{\rho_i}(x_i, x_j) < 0 \text{ for all } j \neq i.$$

Hence, x_i is closer to far-out points on the geodesic corresponding to ρ_i.

Applications I-II: Proof

Corollary: (of Theorems I-III) Every shift-invariant measure on families of geodesics that **do not cross** is supported on families of geodesics containing at most four disjoint paths.

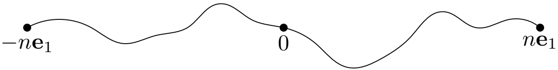


Solution to the midpoint problem

Theorem: (A.-Hoffman) For diverging sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$

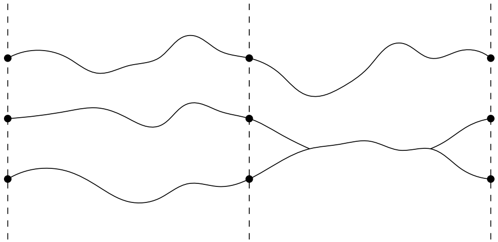
$$\mathbb{P}\big(0 \in \mathrm{Geo}(u_n, v_n)\big) \to 0 \quad \text{as } n \to \infty.$$

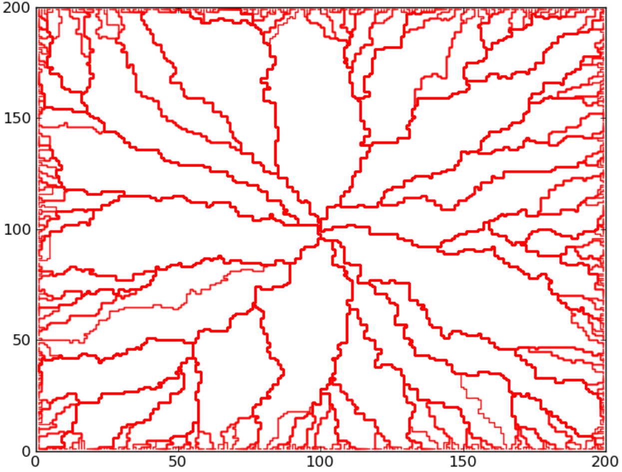
Suppose: $\limsup_{n\to\infty} \mathbb{P}\big(0 \in \mathrm{Geo}(-n\mathbf{e}_1, n\mathbf{e}_1)\big) > \delta$, derive contradiction.



Solution to the midpoint problem

- Consider vertical translates of this event.
- Construct a family of finite non-crossing geodesics.
- The construction induces a measure on non-crossing geodesics.
- Average and take a weak limit. The limiting measure is shift-invariant and supported on infinite non-coalescing geodesics.

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