estados coerentes e distribuições binomiais deformadas

evaldo m f curado
centro brasileiro de pesquisas físicas - cbpf
colmea - maio 2018

resumo

- motivação física
- computação quântica com estados não-ortogonais
- estados coerentes não-lineares
- deformações da distribuição binomial
- deformação assimétrica
- deformação simétrica
- volta à computação quântica

coherent states

Erwi

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The Continuous Transition from Micro- to Macro-Mechanics

(*Die Naturwissenschaften*, 28, pp. 664-666, 1926)

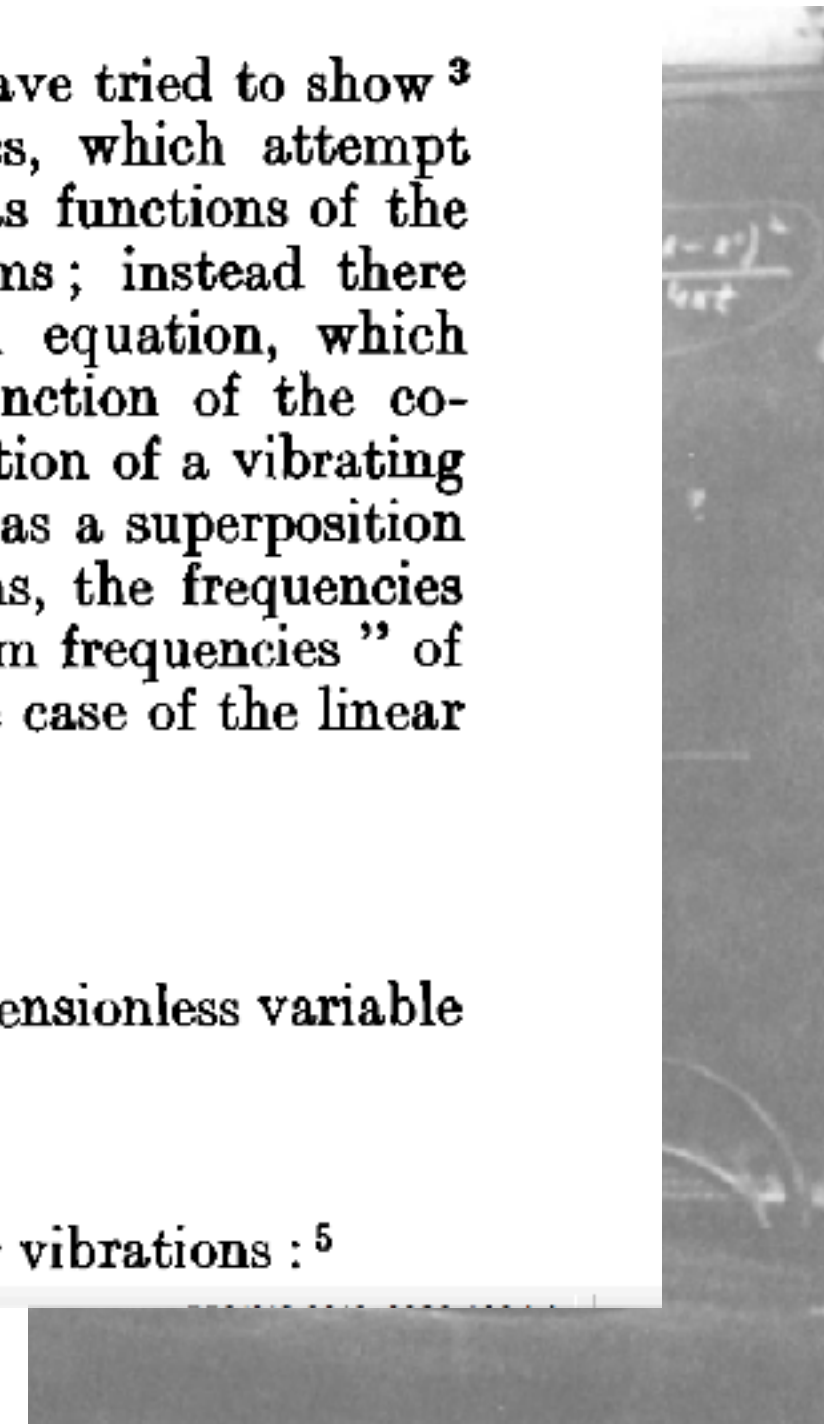
BUILDING on ideas of de Broglie [1] and Einstein,[2] I have tried to show [3] that the usual differential equations of mechanics, which attempt to define the co-ordinates of a mechanical system as functions of the time, are no longer applicable for "small" systems; instead there must be introduced a certain *partial* differential equation, which defines a variable ψ ("wave function") as a function of the co-ordinates and the time. As in the differential equation of a vibrating string or of any other vibrating system, ψ is given as a superposition of pure time harmonic (*i.e.* "sinusoidal") vibrations, the frequencies of which agree exactly with the spectroscopic "term frequencies" of the micro-mechanical system. For example, in the case of the linear Planck oscillator [4] where the energy function is

$$\frac{m}{2}\left(\frac{dq}{dt}\right)^2 + 2\pi^2\nu_0^2 mq^2, \tag{1}$$

when we put, instead of the displacement q, the dimensionless variable

$$x = q \, . \, 2\pi\sqrt{\frac{m\nu_0}{h}}, \tag{2}$$

we get ψ as the superposition of the following proper vibrations: [5]



[4]) d. i. ein Massenpunkt von der Masse m, der, auf   Realteil zu nehmen.

Citations history for 1926NW.....14..664S from the ADS Databases

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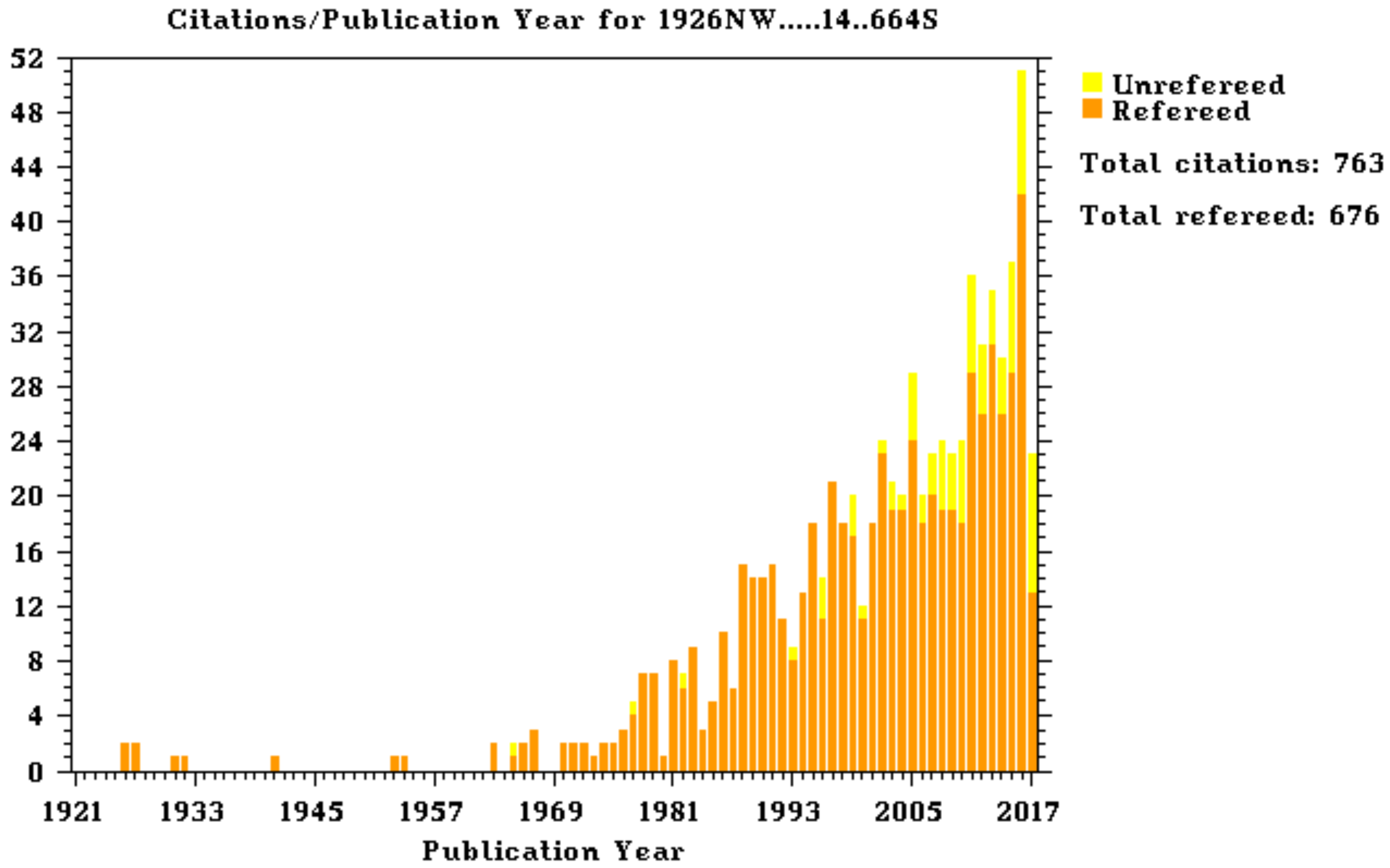


coherent states

Harmonic oscillator

$$H_{HO} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(a^\dagger a + 1/2)$$

$$N = a^\dagger a$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$\longrightarrow \quad [\hat{x}, \hat{p}] = i\hbar \quad \longrightarrow$$

$$[N, a^\dagger] = a^\dagger$$

$$[N, a] = -a$$

$$[a, a^\dagger] = 1$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$N|n\rangle = n|n\rangle$$

$$(n \geq 0)$$

$$|z\rangle = e^{\frac{-|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle \qquad z \in \mathbb{C}$$

$$|z\rangle = \hat{D}(z)|0\rangle = e^{za^\dagger - z^* a}|0\rangle$$

$$a|z\rangle = z|z\rangle$$

CS are not orthogonal

coherent states - mathematical definition

i) normalizability:

$$\langle z|z\rangle = 1$$

ii) continuity in the label:

$$|z-z'| \to 0, \quad || \, |z\rangle - |z'\rangle \, || \to 0$$

iii) resolution of the identity:

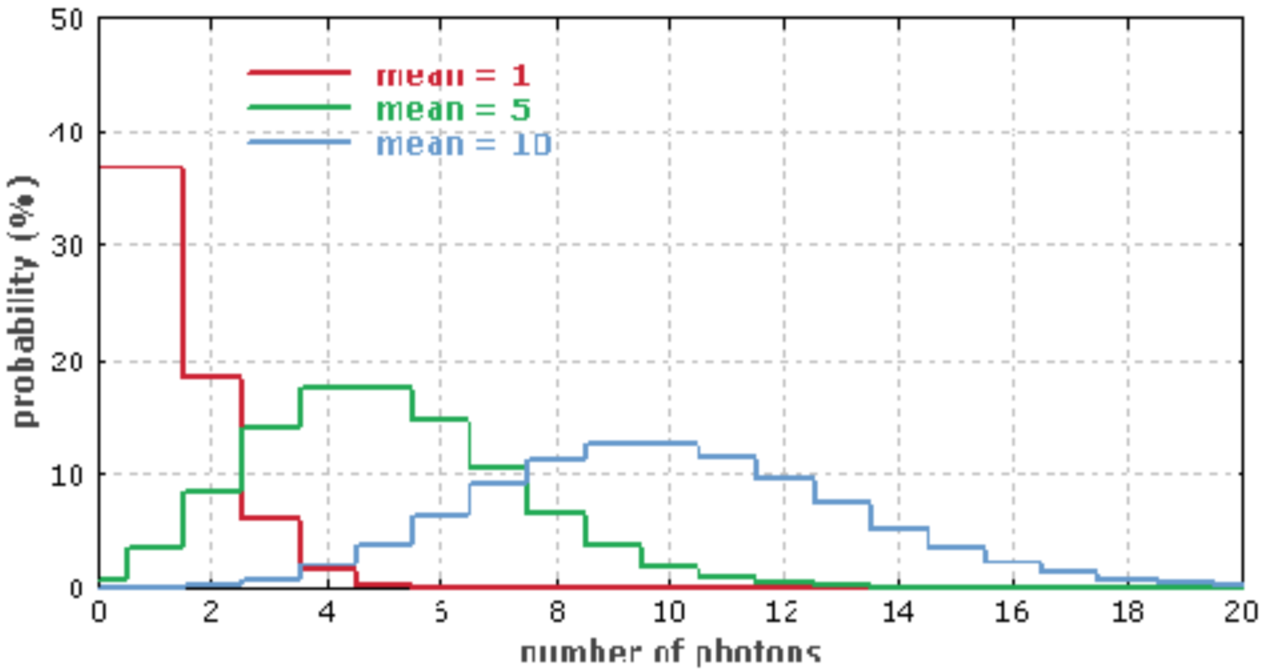
$$\int d^2z \, \omega(z) \, |z\rangle\langle z| = 1$$

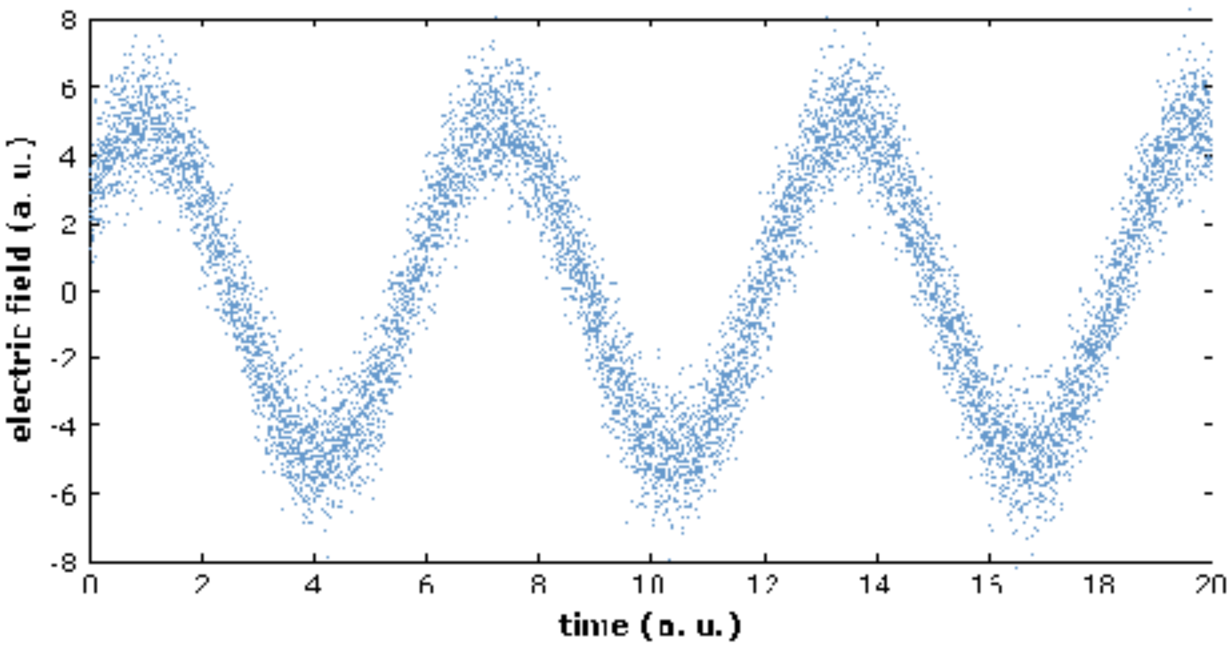
- can be defined for any quantum system

coherent states - applications

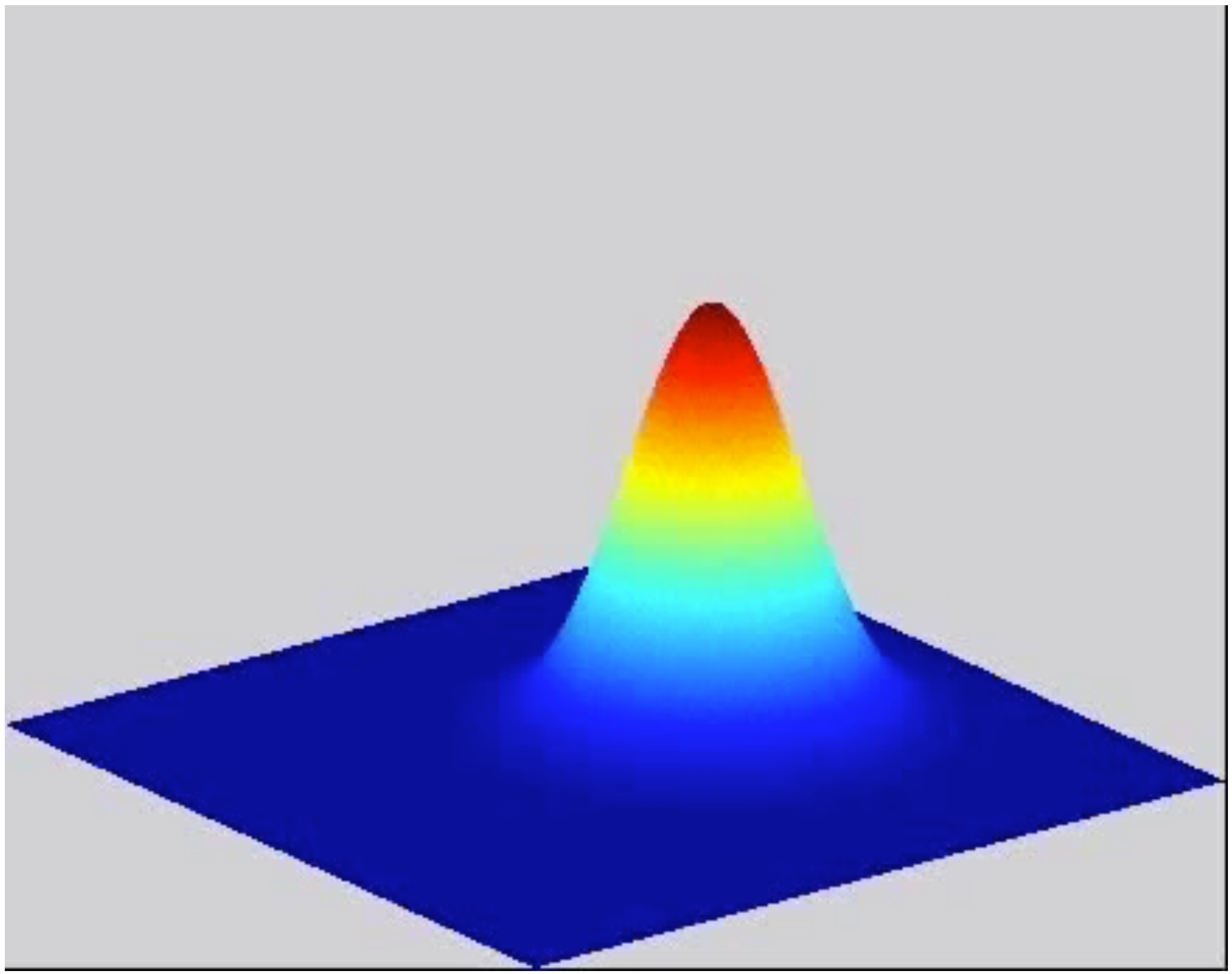
$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$P(n) = |\langle n|\alpha\rangle|^2 = \langle n\rangle^n \frac{\exp(-\langle n\rangle)}{n!} \qquad \langle n\rangle = |\alpha|^2$$





Roy Glauber, Noble Prize for Quantum Optics Theory, 2005. Quantum state of a laser



quantum communication

quantum states for information binary communication

- nonorthogonal quantum states - codewords $(\rho_0 \text{ and } \rho_1)$
- measurements - operators M_0 and M_1 (POVM)

$$p(m_0|\rho_1) = Tr[M_0\rho_1]$$

$$p(m_1|\rho_0) = Tr[M_1\rho_0]$$

$$M_0 + M_1 = \mathbb{I}$$

- error probability

ξ_0 : probability to send the state ρ_0

ξ_1 : probability to send the state ρ_1

$$p(M_0, M_1) = \xi_0\, p(m_1|\rho_0) + \xi_1\, p(m_0|\rho_1)$$

$$p(M_0, M_1) = \xi_0\, p(m_1|\rho_0) + \xi_1\, p(m_0|\rho_1)$$

$$p(M_0, M_1) = \xi_1 \; + Tr[M_1\, \Gamma] \qquad\qquad \Gamma \equiv \xi_0 \rho_0 - \xi_1 \rho_1$$

- minimizing the error in receiver measurement over all possible POVM's (M_0 e M_1) -> Helstrom bound

$$P_H \equiv \min_{M_0, M_1} p(M_0, M_1) = \xi_1 + \min_{M_1} Tr[M_1\, \Gamma]$$

$$\Gamma = \sum_n \lambda_n |\gamma_n\rangle\langle\gamma_n|$$

$$Tr[M_1\, \Gamma] = \sum_n \lambda_n \langle\gamma_n|M_1|\gamma_n\rangle$$

$$P_H = \xi_1 + \sum_{\lambda_n < 0} \lambda_n m_{1,n}$$

- projector operator on all eigenstates $|\lambda_n\rangle$ with negative λ_n

Helstrom bound for coherent states

laser -> $$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \langle n\rangle = |\alpha|^2$$

$$\mathcal{A} = \{|0\rangle, |\alpha\rangle\} \qquad \rho_0 = |0\rangle\langle 0| \qquad \rho_1 = |\alpha\rangle\langle\alpha|$$

superposition -> $$\langle 0|\alpha\rangle = e^{-|\alpha|^2/2} = e^{-\langle n\rangle/2}$$

$$P_H = \xi_1 + \lambda_- = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1|\langle 0|\alpha\rangle|^2}\right)$$

Helstrom bound for perfect detection (efficiency η = 1)

$$P_H = \xi_1 + \lambda_- = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\langle n\rangle}}\right)$$

Helstrom bound for imperfect detection (efficiency η < 1)

probability to detect n-photons using a non-ideal photodetector (η < 1)

$$p_n(\eta) = \sum_{m=n}^{\infty} \binom{m}{n} \eta^n (1-\eta)^{m-n} p_m(\eta = 1)$$

$$p_n(\eta) = \frac{\eta^n |\alpha|^{2n}}{n!} e^{-\eta|\alpha|^2}$$

$$P_H = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\eta\langle n\rangle}}\right)$$

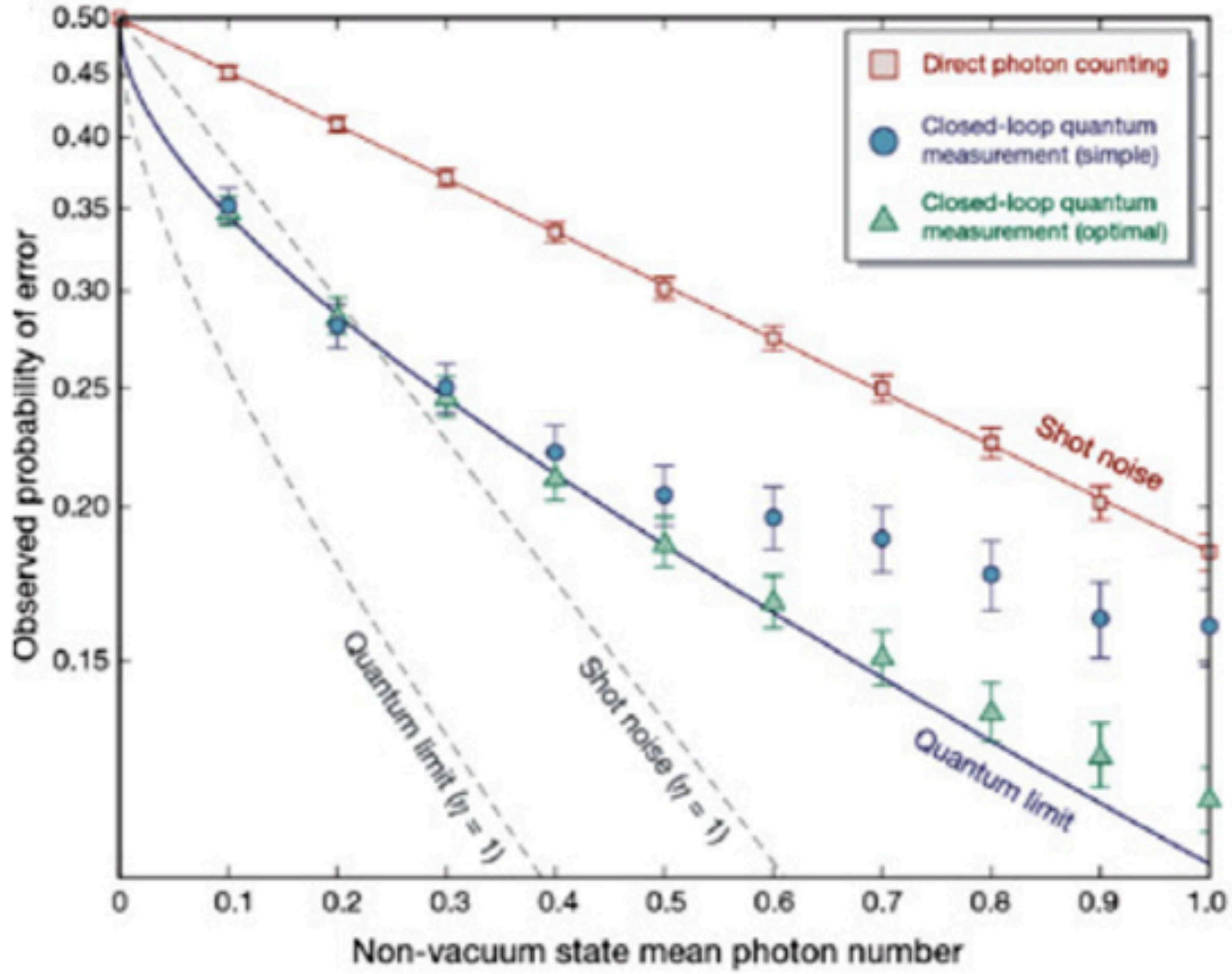


Figure 2. Measured probability of error versus mean photon number for both direct photon counting (red squares) and the CMG closed-loop measurement interpreted using a Bayesian estimator that assumes application of the optimal closed-loop control policy (blue circles) and one that accounts for experimental imperfections (green triangles). All data points were obtained from ensembles of 100 000 measurement trajectories, with error bars that reflect the sample standard deviation.

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Nonlinear coherent states for optimizing quantum information

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Abstract
Part of the difficulties in implementing communication in quantum information stems from the fragility of Schrödinger's cat-like superpositions. A recent experiment in quantum optics by Cook *et al* (2007 *Nature* **446** 774) has proved the feasibility of a feedback-mediated quantum measurement for discriminating between optical coherent states under photodetection. Minimizing the error in receiver measurement over all possible POVMs leads to the so-called quantum error probability or 'Helstrom bound', and CMG measurements validate the theoretical prediction by Helstrom, Dolinar and Geremia concerning this bound. In this work, we present some preliminary theoretical and numerical explorations concerning the properties of the Helstrom bound in binary (or multibinary) communication involving non-Poissonian or nonlinear coherent states.

- Glauber coherent states -> Poissonian number distribution
- real lasers -> better discribed by states that lead to almost or non-Poissonian distributions

- binomial distribution -> Poisson distribution
- deformed binomial distribution -> non-Poissonian distribution

4. NLCSs for binary communication

We now turn to families of states for a one-mode electromagnetic quantum field that have the following form in the corresponding Fock space:

$$|\alpha; \mathcal{X}\rangle \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{\sqrt{\mathcal{N}(|\alpha|^2)}} \frac{\alpha^n}{\sqrt{x_n!}} |n\rangle. \tag{12}$$

State $|n\rangle$ is a number eigenstate, i.e. an eigenstate of the photon number operator $N = a^\dagger a$, $N|n\rangle = |n\rangle$. The 'factorial' $x_n!$ means

$$x_n! = x_1\, x_2\, \cdots x_n, \quad x_0! \stackrel{\text{def}}{=} 1, \tag{13}$$

where the x_n's, $n = 1, 2, \ldots,$ form, with $x_0 \equiv 0$, the sequence of positive numbers,

$$\mathcal{X} \stackrel{\text{def}}{=} \{x_0 = 0,\ x_1,\ x_2, \ldots, x_n, \ldots\}.$$

States (12), denoted more simply by $|\alpha\rangle$ in the sequel, are normalized, $\langle\alpha|\alpha\rangle = 1$, which means that the function $\mathcal{N}$ appearing in their expression reads as the 'exponential' associated with the factorials (13):

$$\mathcal{N}(\mathrm{t}) = \sum_{n=0}^{\infty} \frac{\mathrm{t}^n}{x_n!}. \tag{14}$$

Helstrom bound for nonlinear coherent states

$$|\alpha;\chi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\mathcal{N}(|\alpha|^2)}} \frac{\alpha^n}{\sqrt{x_n!}} |n\rangle$$

$$\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!}$$

$$\mathcal{A} = \{|0\rangle, |\alpha;\chi\rangle\}$$

$$\langle 0|\alpha;\chi\rangle = \frac{1}{\sqrt{\mathcal{N}(t)}}$$

$$n \mapsto |\langle n|\alpha\rangle|^2 = \frac{1}{\mathcal{N}(|\alpha|^2)} \frac{|\alpha|^{2n}}{x_n!}$$

Helstrom bound for perfect detection (efficiency η = 1)

$$P_H^\chi = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1 \frac{1}{\mathcal{N}(t)}}\right)$$

$$\langle n\rangle = t\frac{d}{dt}\ln \mathcal{N}(t)$$

Mandel parameter

$$Q_M = \frac{\langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle} = \frac{(\Delta n)^2}{\langle n \rangle} - 1$$

Poisson $\quad (t = |\alpha|^2)$

$$\langle n \rangle = t$$

$$\langle (\Delta n)^2 \rangle = t$$

$$Q_M = 0 \ \text{(Poisson)} \quad \rightarrow \quad \begin{cases} \text{sub-Poissonian} \quad Q_M < 0 \\ \text{super-Poissonian} \ \ Q_M > 0 \end{cases}$$

independent systems

uncorrelated system - binomial distribution

- n independent trials with two possible outcomes - "win" or "loss"

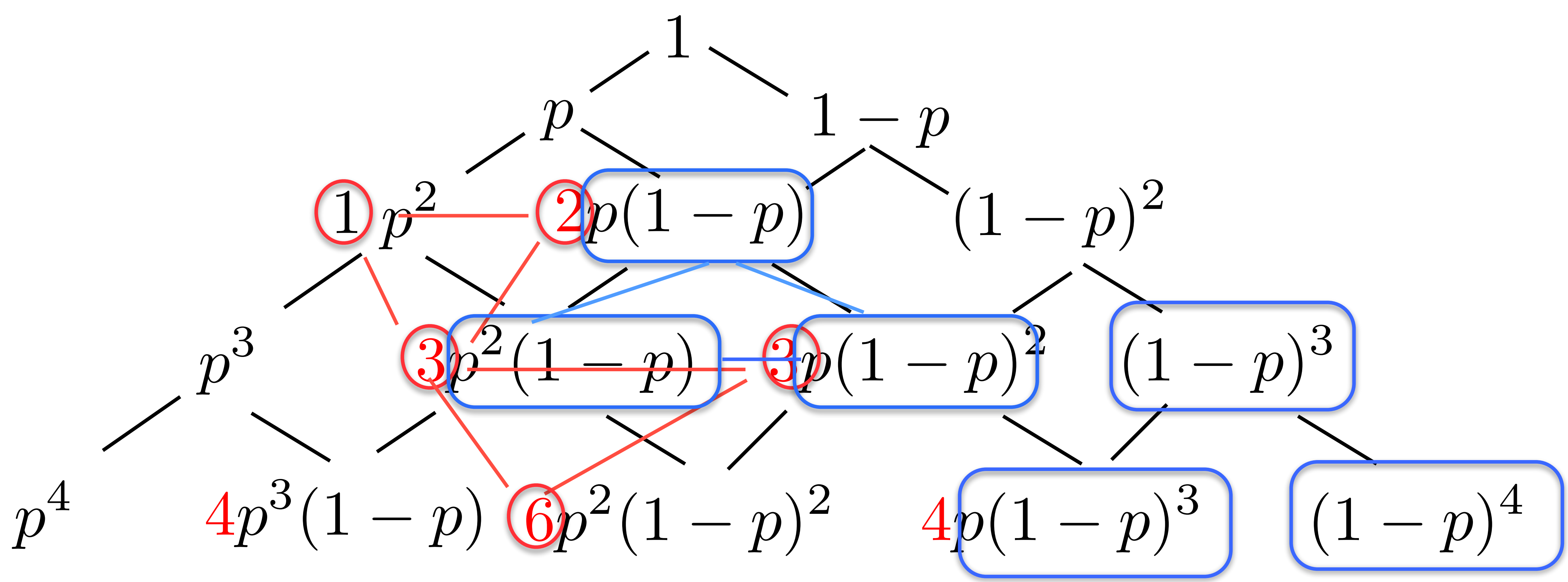
$$\eta \in [0,1]$$

$$p_k^{(n)}(\eta) = \begin{pmatrix} n \\ k \end{pmatrix} \eta^k \, (1-\eta)^{n-k} = \frac{n!}{(n-k)!\,k!} \, \eta^k \, (1-\eta)^{n-k}$$

probability to have k wins in n trials - regardless the order

$$\varpi_k^{(n)} = \eta^k (1-\eta)^{n-k} \quad \Rightarrow \quad \varpi_k^{(n-1)} = \varpi_k^{(n)} + \varpi_{k+1}^{(n)}$$

Leibniz triangle rule



Pascal and Leibniz rules

η -> 0 - Poisson

n is large and n η -> t (cte)

$$p_k^{(n)}(\eta) = \begin{pmatrix} n \\ k \end{pmatrix} \eta^k \, (1-\eta)^{n-k} = \frac{n!}{(n-k)!\,k!}\,\eta^k \, (1-\eta)^{n-k}$$

$$= \frac{n.(n-1)...(n-k+1)}{k!}\left(\frac{t}{n}\right)^k \left(1-\frac{t}{n}\right)^{n-k}$$

$$\sim \frac{n^k}{k!}\left(\frac{t}{n}\right)^k \left(1-\frac{t}{n}\right)^{n-k}$$

$$\sim \frac{t^k\,e^{-t}}{k!}$$

$$p_k^{(n)}(\eta) \sim \frac{t^k\,e^{-t}}{k!}$$

p (η) finite - Gaussian

n, n η , n (1-η) are large

$$\binom{n}{k} \to_{k=nx} \binom{n}{nx} \sim \frac{1}{\sqrt{2\pi n x(1-x)}} \exp\left[n\left(-x\log[x]-(1-x)\log[1-x]\right)\right]$$

$$\eta^{nx}(1-\eta)^{n(1-x)} \to \exp\left[n\left(x\log\left[\eta\right]+(1-x)\log\left[1-\eta\right]\right)\right]$$

$$\binom{n}{k}\eta^k(1-\eta)^{n-k} \sim \frac{1}{\sqrt{2\pi n x(1-x)}} \exp\left[n\left(\underline{-x\log\left[\frac{x}{\eta}\right]-(1-x)\log\left[\frac{1-x}{1-\eta}\right]}\right)\right]$$

$$\sum_{k=0}^{n}\binom{n}{k}\eta^k(1-\eta)^{n-k} \to_{k=nx} n\int_0^1 dx \frac{1}{\sqrt{2\pi n x(1-x)}} \exp[nf(x,\eta)]$$

$$f(x,\eta) = -x\log\left[\frac{x}{\eta}\right]-(1-x)\log\left[\frac{1-x}{1-\eta}\right]$$

$$f'(x^\star,\eta)=0 \Rightarrow x^\star=\eta \qquad f''(x,\eta)|_{x=\eta} = -\frac{1}{x(1-x)}|_{x=\eta} = -\frac{1}{\eta(1-\eta)}$$

$$f(x,\eta) \simeq \overset{0}{f(x^\star=\eta,\eta)} + \overset{0}{f'(x^\star=\eta,\eta)}(x-\eta) + \frac{1}{2}f''(x^\star=\eta,\eta)(x-\eta)^2 + \ldots$$

$$\sum_{k=0}^{n} \binom{n}{k} \eta^k (1-\eta)^{n-k} \to_{k=nx} n \int_0^1 dx \frac{1}{\sqrt{2\pi n x(1-x)}} \exp[n f(x,\eta)]$$

$$(k = nx)$$

Laplace's method $\to \dfrac{n}{\sqrt{2\pi n \eta(1-\eta)}} \displaystyle\int_{-\infty}^{\infty} dx \exp\left[-\frac{n}{2\eta(1-\eta)}(x-\eta)^2\right] = 1$

$$\Rightarrow \mathcal{P}_k^{(n)} = \binom{n}{k} \eta^k (1-\eta)^{n-k} \to \frac{1}{\sqrt{2\pi n \eta(1-\eta)}} \exp\left[-\frac{n}{2\eta(1-\eta)}(x-\eta)^2\right]$$

limit distribution is a Gaussian

binomial case - S_{BG}

$$\mathfrak{P}_k^{(n)} = \binom{n}{k} \eta^k (1-\eta)^{n-k} \qquad \eta \in [0,1] \qquad \sum_{k=0}^{n} \mathfrak{P}_k^{(n)} = 1$$

$$\varpi_k^{(n)} = \frac{\mathfrak{P}_k^{(n)}}{\binom{n}{k}} = \eta^k (1-\eta)^{n-k} \qquad \sum_{k=0}^{n} \binom{n}{k} k \varpi_k^{(n)}(\eta) = n\eta =< k >$$

$$S_{BG}(\eta) = -\sum_{k=0}^{n} \binom{n}{k} \varpi_k^{(n)} \log(\varpi_k^{(n)}) = -\sum_{k=0}^{n} \mathfrak{P}_k^{(n)} \log(\varpi_k^{(n)})$$

$$S_{BG}(\eta) = -(< k > \log \eta + < n-k > \log(1-\eta))$$

$$= -n \left[\eta \log \eta + (1-\eta)\log(1-\eta)\right] \leq S_{BG}(1/2) = n \log 2$$

S_{BG} is extensive

binomial case - Rényi entropy

$$S_R^{(q)}(\eta) = \frac{1}{1-q} \log \left[\sum_{k=0}^{n} \binom{n}{k} \left(\varpi_k^{(n)} \right)^q \right]$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(\varpi_k^{(n)}(\eta) \right)^q = \sum_{k=0}^{n} \binom{n}{k} \eta^{qk} (1-\eta)^{q(n-k)} = (\eta^q + (1-\eta)^q)^n$$

$$= \exp\left[n \log\left(\eta^q + (1-\eta)^q \right) \right]$$

$$S_R^{(q)}(\eta) = \frac{n}{1-q} \log\left[\eta^q + (1-\eta)^q \right] \leq S_R^{(q)}(1/2) = n \log 2$$

$$(0 < q < 1)$$

S_R is extensive as well!

correlation -> deformation

deformations and correlations

- in most of realistic models in physics one must take correlations into account: events which are usually presented as independent, like in a binomial Bernoulli process, are actually submitted to correlative perturbations.
- these perturbations lead to deformations of the mathematical independent laws.
- for instance, the deformation of the Poisson distribution upon which is based the construction of Glauber coherent states in quantum optics leads to the so-called nonlinear coherent states. The realization of a special class of these states, adapted to this deformation, has been proposed in the quantized motion of a trapped atom in a Paul trap.

deformed binomial distribution =>
correlation between events

Laplace - de Finetti modification of the binomial law

Laplace (1774)

$$\mathfrak{P}_k^{(n)} = \binom{n}{k} \varpi_k^{(n)}$$

binomial $\rightarrow \varpi_k^{(n)} = \eta^k (1-\eta)^{n-k}$

de Finetti (1937)

$$\widetilde{\mathfrak{P}}_k^{(n)} := \binom{n}{k} \widetilde{\varpi}_k^{(n)}$$

binary correlated system

$$\widetilde{\varpi}_k^{(n)} := \int_0^1 dy\, y^k (1-y)^{n-k} g(y) \quad \text{where} \quad \int_0^1 dy\, g(y) = 1$$

- $\widetilde{\varpi}_k^{(n-1)} = \widetilde{\varpi}_k^{(n)} + \widetilde{\varpi}_{k+1}^{(n)}$ Leibniz triangle rule
- binary exchangeable stochastic process

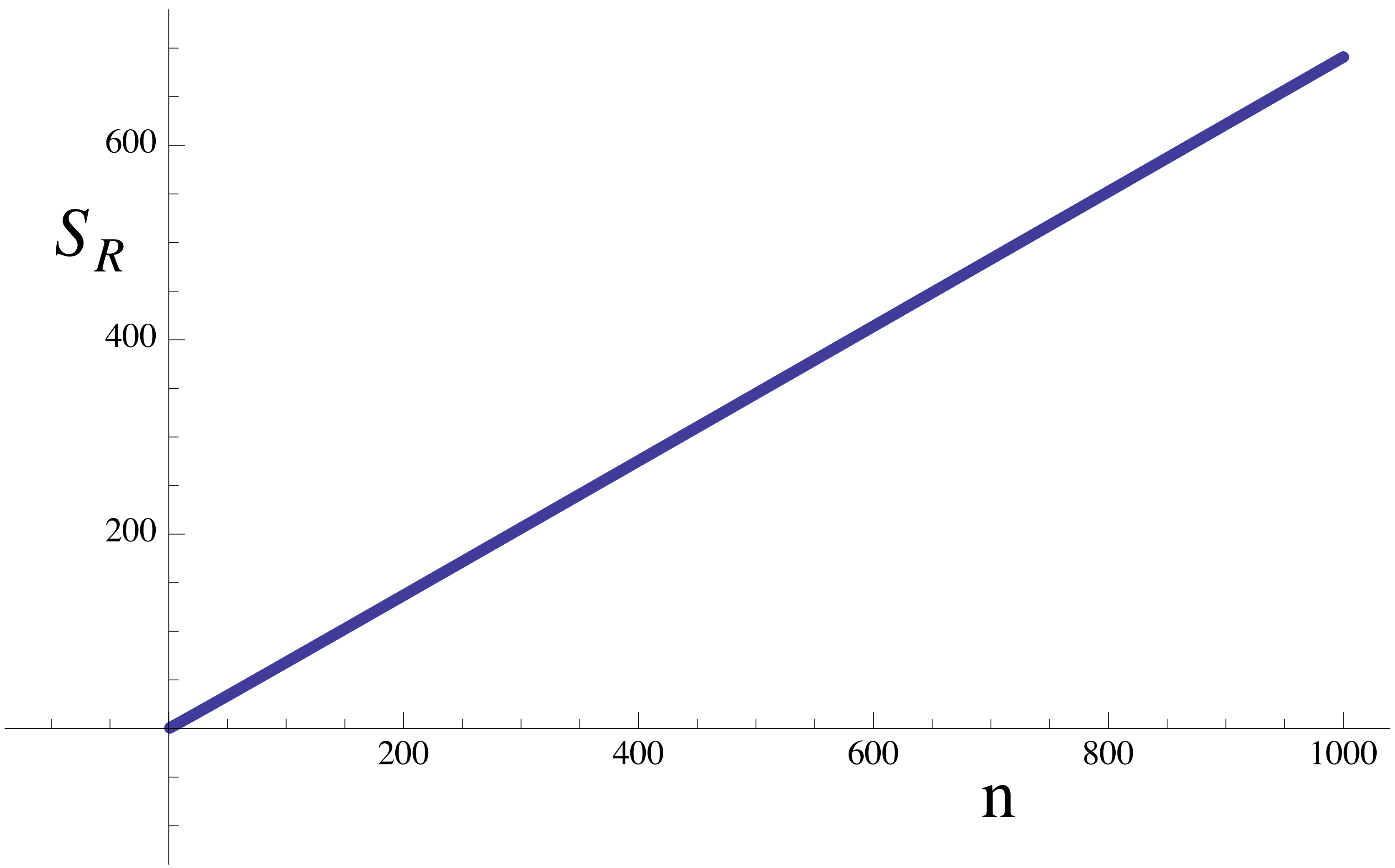
limit distribution and extensivity

- $\rho(x) = g(x)$
- Boltzmann-Gibbs entropy is extensive

Hanel, Thurner, Tsallis, EPJB (2009)

Rényi

$$\widetilde{S}_R^{(q)}[g] = \frac{1}{1-q} \log \left[\sum_{k=0}^{n} \binom{n}{k} \left(\widetilde{\varpi}_k^{(n)} \right)^q \right]$$



$$\widetilde{S}_R^{(q)}[g] \sim n \log 2$$

$$g(y) = \frac{8}{\pi} \sqrt{y(1-y)}$$

$$q = 1/2$$

two microscopic entropies are extensive for the Laplace-de Finetti case

H Bergeron, EMFC, JP Gazeau, Ligia MCS Rodrigues, Physica A 441 (2016) 23 "more correlated" systems

correlated events - deformed binomial

- n correlated trials with two possible outcomes - "win" or "loss"

$$x_0, x_1, x_2, x_3, \cdots, x_n, \cdots \qquad x_0 = 0 \qquad x_n > x_{n-1}$$

$$x_n! = x_1 x_2 \cdots x_n, \quad x_0! \equiv 1$$

$$\mathfrak{p}_k^{(n)}(\eta) = \frac{x_n!}{x_{n-k}! x_k!} q_k(\eta) q_{n-k}(1-\eta)$$

deformed probability to have k wins in n trials - induced by correlations

$$\forall n \in \mathbb{N}, \quad \sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = 1. \qquad (x_n \to n \Rightarrow q_k \to \eta^k)$$

probabilistic interpretation

$q_k(\eta)$ has to be nonnegative for all $\eta \in [0,1]$

program

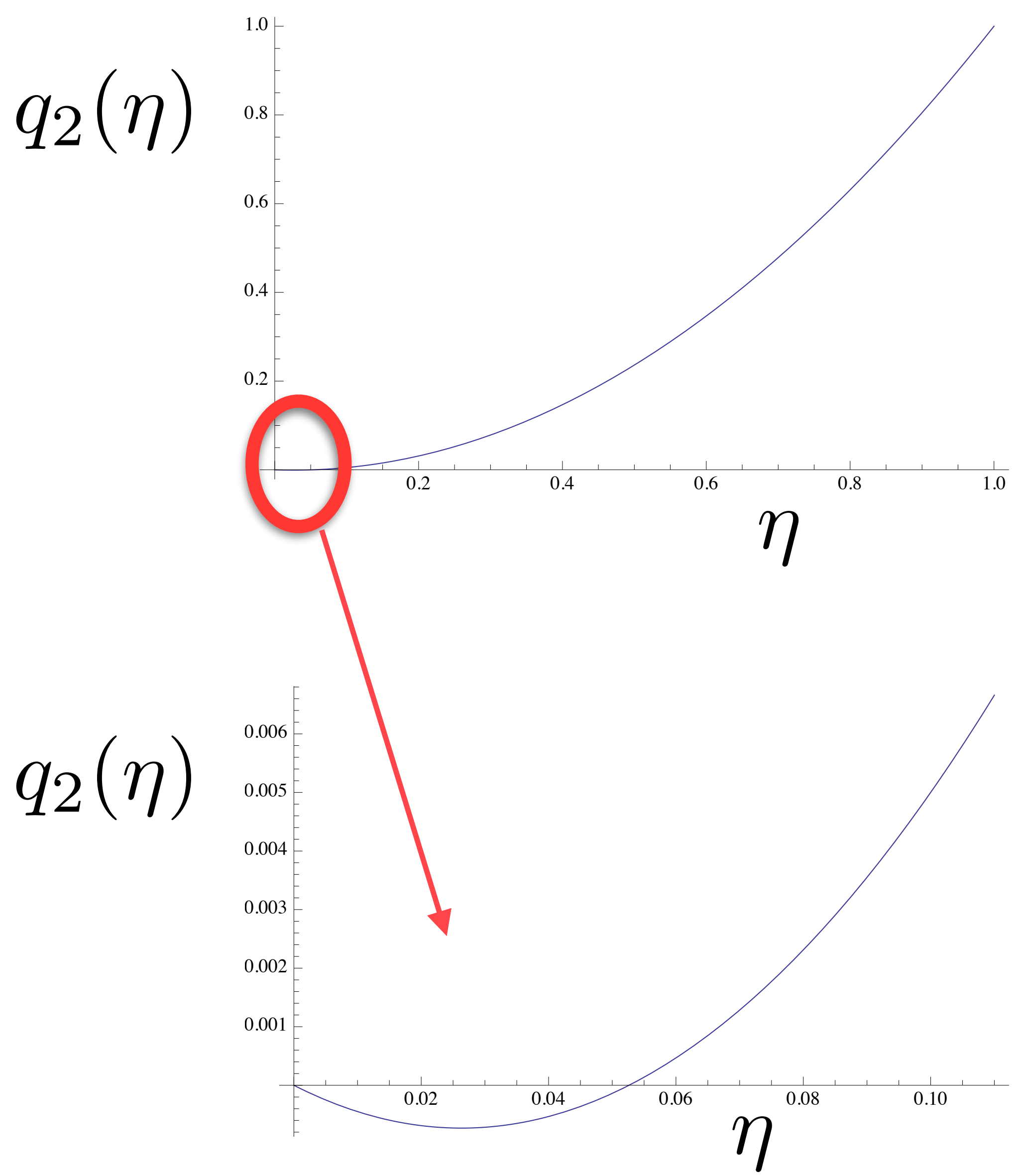
- choose a sequence: $x_0, x_1, x_2, \ldots, x_n, \ldots$
- construct the generalized exponential N(t)
- construct the functions $q_n(\eta)$ using

$$q_n(\eta) + q_n(1-\eta) = \sum_{k=0}^{n-1} \binom{n}{k} q_k(\eta) q_{n-k}(1-\eta) \qquad \begin{aligned} q_0(\eta) &= 1 \\ q_1(\eta) &= \eta \end{aligned}$$

- construct the function $p_k^{(n)}(\eta)$
- this procedure does not work in general!

example of the wrong route

$$x_0 = 0,\, x_1 = 1 - \epsilon,\, x_2 = 2 - \epsilon, \cdots, x_n = n - \epsilon, \cdots$$



$\epsilon = 0.1$

solving the positiveness problem by means of generating functions

$$\mathfrak{p}_k^{(n)}(\eta) = \frac{x_n!}{x_k!\,x_{n-k}!} q_k(\eta) q_{n-k}(1-\eta)$$

$\eta \to 1-\eta \qquad k \to n-k \quad \mapsto$ symmetric distributions

$$\sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = 1 \qquad \left(\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!}\right)$$

$$\forall n, k \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \mathfrak{p}_k^{(n)}(\eta) \geq 0$$

$$G(\eta; t) := \sum_{n=0}^{\infty} \frac{q_n(\eta)}{x_n!} t^n \qquad \left(\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!} \right)$$

$$\sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = 1$$

$$\boxed{\mathfrak{p}_k^{(n)}(\eta) = \frac{x_n!}{x_{n-k}! x_k!} q_k(\eta) q_{n-k}(1-\eta)}$$

$$\longrightarrow G(\eta; t)\, G(1-\eta; t) = \mathcal{N}(t)$$

$$\mathcal{N}(t) = \left(\sum_{k=0}^{\infty} \frac{q_k(\eta) t^k}{x_k!} \right) \left(\sum_{m=0}^{\infty} \frac{q_m(1-\eta) t^m}{x_m!} \right)$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_k(\eta)}{x_k!} \frac{q_m(1-\eta)}{x_m!} t^{k+m} \qquad k + m \to n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{x_n!}{x_k! x_{n-k}!} q_k(\eta) q_{n-k}(1-\eta) \right) \frac{t^n}{x_n!}$$

$$\forall n,k \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \mathfrak{p}_k^{(n)}(\eta) \geq 0$$

$$G(\eta;t) := \sum_{n=0}^{\infty} \frac{q_n(\eta)}{x_n!} t^n \qquad\qquad G(\eta;t)\, G(1-\eta;t) = \mathcal{N}(t)$$

$$G(\eta;t) = \pm\sqrt{\mathcal{N}(t)}e^{\Phi(\eta,1-\eta;t)} \qquad\qquad \left(\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!}\right)$$

$$\Phi(x,y;t) = -\Phi(y,x;t)$$

simplest case: $\Phi(x,y;t) = (x-y)\varphi(t)$

$$G(0;t) = 1 \Leftrightarrow G(1;t) = \mathcal{N}(t)$$

$$\mathcal{N}(t) = e^{2\Phi(1,0;t)} = e^{2\varphi(t)} \qquad\qquad G(\eta;t) = e^{(2\eta)\varphi(t)} = (\mathcal{N}(t))^\eta$$

$$\forall \eta \in [0,1], \quad \mathcal{N}(t)^\eta = \sum_{n=0}^{\infty} q_n(\eta) \frac{t^n}{x_n!}$$

definitions

definition 1: Σ_0 is the set of entire series $f(z) = \Sigma_{n=0}\ a_n\ z^n$ possessing a non-vanishing radius of convergence and verifying the conditions
$a_0 = 0$, $a_1 > 0$ and $\forall\ n \geq 2$, $a_n \geq 0$.

Lemma 2.1. *$\forall a, b > 0$, $\forall \alpha \in [0,1]$, the functions F defined as $F(t) = e^{at} - 1$, $F(t) = -a \ln(1 - bt)$, $F(t) = a \ln \dfrac{1 + \alpha bt}{1 - bt}$ belong to Σ_0.*

Proof. It is straightforward to show that the series expansions verify the required conditions.

□

Some simple properties of Σ_0 are interesting for our purpose:

Proposition 2.1. *Properties of Σ_0*

1. *$\forall F, G \in \Sigma_0, F + G \in \Sigma_0$,*
2. *Σ_0 is a convex set,*
3. *$\forall F \in \Sigma_0$, $\forall \eta \in [-1, 1[$, $t \mapsto F(t) - F(\eta t) \in \Sigma_0$ and $t \mapsto F(t) + F(-\eta t) \in \Sigma_0$,*
4. *$\forall F, G \in \Sigma_0$, $\forall a > 0$, $(a + F)G \in \Sigma_0$,*
5. *$\forall F, G \in \Sigma_0$, $F \circ G \in \Sigma_0$.*

definition 2: Σ is the set of entire series $N(t) = \Sigma_{n=0}\, a_n\, t^n$ possessing a non-vanishing radius of convergence and verifying the conditions $a_0 = 1$, and $\forall\ n \geq 1,\ a_n > 0$.

Proposition 2.2. *The set Σ verifies the following properties:*

1. $\forall \mathcal{N}_1, \mathcal{N}_2 \in \Sigma$, $\mathcal{N}_1 + \mathcal{N}_2 - 1 \in \Sigma$,
2. Σ *is a convex set,*
3. $\forall \mathcal{N}_1, \mathcal{N}_2 \in \Sigma$, $\mathcal{N}_1 \mathcal{N}_2 \in \Sigma$,
4. $\forall F \in \Sigma_0, \quad t \mapsto e^{F(t)} \in \Sigma$.

definition 3: Σ_+ is the set of entire series $N(t) = \Sigma_{n=0}\, a_n\, t^n$ possessing a non-vanishing radius of convergence, verifying the conditions
$a_0 = 1$, and $\forall\, n \geq 1$, $a_n > 0$ and satisfying $N(t) = \exp[F(t)]$ where $F(t)$ belongs to the set Σ_0

main theorem:

$$\Sigma_+ = \{\mathcal{N} \in \Sigma \,|\, \forall \eta \in [0,1[,\ q_n(\eta) > 0\} = \left\{e^F \,|\, F \in \Sigma_0\right\}$$

Proposition 2.3. *The set Σ_+ defined in Eq.(16) satisfies the following properties:*

1. $\forall \mathcal{N}_1, \mathcal{N}_2 \in \Sigma_+,\ \mathcal{N}_1\mathcal{N}_2 \in \Sigma_+$,
2. $\forall F \in \Sigma_0, \quad t \mapsto e^{F(t)} \in \Sigma_+$.

solving the positiveness problem by means of generating functions

$$\mathfrak{p}_k^{(n)}(\eta) = \frac{x_n!}{x_k!\, x_{n-k}!} q_k(\eta) q_{n-k}(1-\eta)$$

$\eta \to 1-\eta \qquad k \to n-k \quad \mapsto$ symmetric distributions

$$\sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = 1 \qquad G_{\mathcal{N},\eta}(t) = \mathcal{N}(t)^{\eta} = \sum_{n=0}^{\infty} \frac{q_n(\eta)}{x_n!} t^n$$

Σ → $\mathcal{N}(t) = 1 + a_1 t + a_2 t^2 + \cdots$, all $a_n > 0,\ n \geq 1$

Σ_0 → $(F(t) = a_1 t + a_2 t^2 + \cdots)$, $a_1 > 0$, $a_2, \ldots \geq 0$

main theorem:

$$\Sigma_+ = \{\mathcal{N} \in \Sigma \,|\, \forall \eta \in [0,1[,\ q_n(\eta) > 0\} = \left\{ e^F \,|\, F \in \Sigma_0 \right\}$$

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Generating functions for generalized binomial distributions

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Symmetric generalized binomial distributions

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example 1 - q-Gaussian

$$\boxed{\mathcal{N}(t) = \left(1 - \frac{t}{\alpha}\right)^{-\alpha}}, \quad \alpha > 0 \qquad \left(\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!}\right)$$

$$x_n! = \alpha^n \frac{\Gamma(\alpha) n!}{\Gamma(n+\alpha)} = \frac{\alpha^n n!}{(\alpha)_n} \qquad \boxed{x_n = \frac{n\alpha}{n+\alpha-1}}, \qquad \lim_{n\to\infty} x_n = \alpha$$

$$q_n(\eta) = \frac{\Gamma(\alpha)}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha\eta)}{\Gamma(\alpha\eta)} = \frac{(\alpha\eta)_n}{(\alpha)_n}$$

$$q_0(\eta) = 1 \qquad q_1(\eta) = \eta$$

$$\boxed{\mathfrak{p}_k^{(n)}(\eta) = \frac{x_n!}{x_{n-k}! x_k!} q_k(\eta) q_{n-k}(1-\eta)}$$

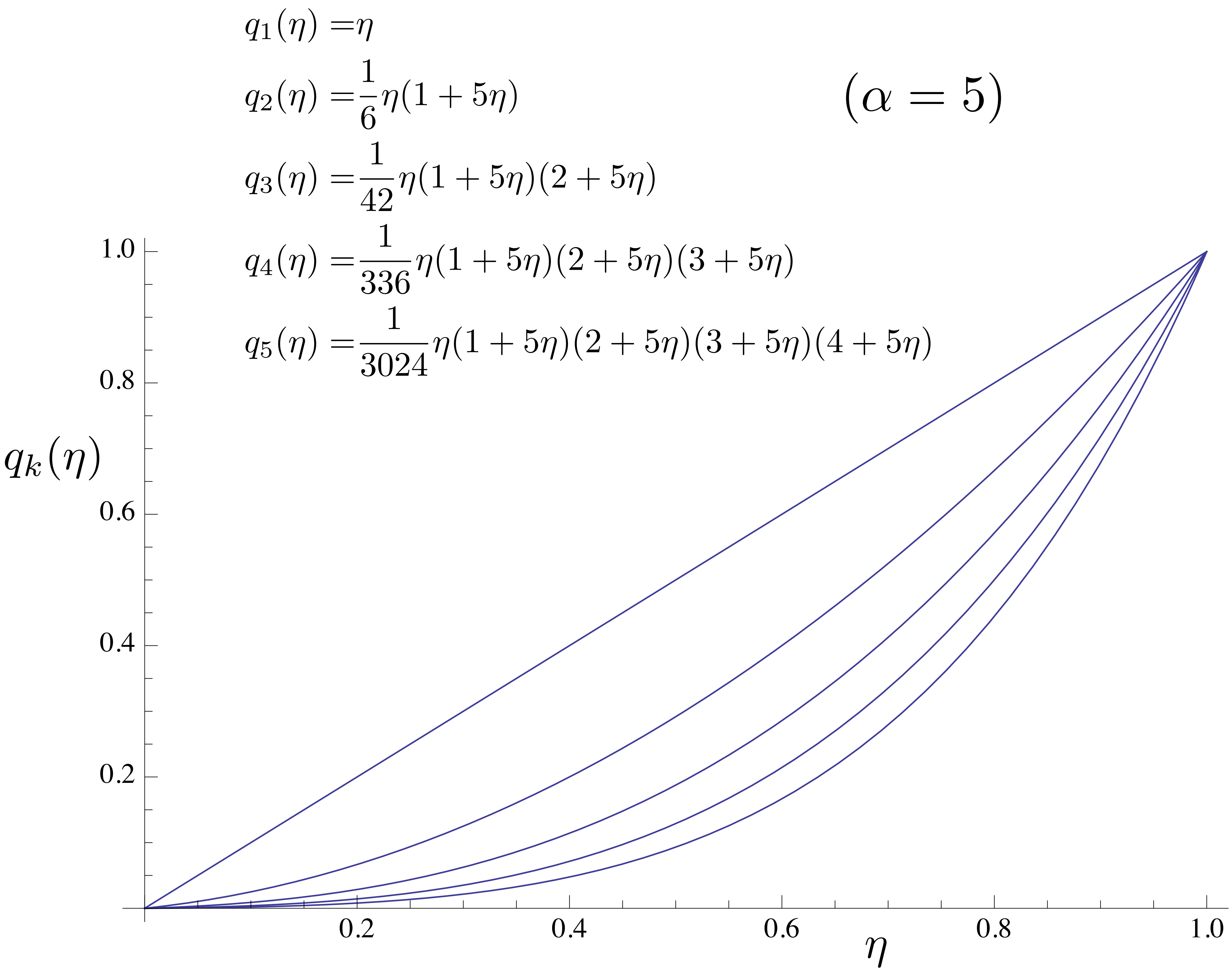
$$\mathfrak{p}_k^{(n)}(\eta) = \binom{n}{k} \frac{\Gamma(\alpha)}{\Gamma(\eta\alpha)\Gamma((1-\eta)\alpha)} \frac{\Gamma(\eta\alpha + k)\Gamma((1-\eta)\alpha + n - k)}{\Gamma(\alpha + n)}$$

Pólya-Markov distribution

G. Pólya (1923): urn scheme. From a set of **b** blue balls and **r** red balls contained in an urn one extracts one ball and return it to the urn together with **c** balls of the same color. The probability to select in the urn **k** blue balls after the n-th trial is given by $\mathfrak{p}_k^{(n)}(\eta)$ with

$$\eta = \frac{b}{b+r} \qquad \alpha = \frac{b+r}{c}$$

(special cases: $c = 0$; $c = -1$)


$q_1(\eta) = \eta$
$q_2(\eta) = \frac{1}{6}\eta(1+5\eta)$
$q_3(\eta) = \frac{1}{42}\eta(1+5\eta)(2+5\eta)$
$q_4(\eta) = \frac{1}{336}\eta(1+5\eta)(2+5\eta)(3+5\eta)$
$q_5(\eta) = \frac{1}{3024}\eta(1+5\eta)(2+5\eta)(3+5\eta)(4+5\eta)$
$(\alpha = 5)$
$q_k(\eta)$
η
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0

asymptotic behavior at large n

$$\mathfrak{p}^n_{k=nx} \sim \frac{1}{n} \frac{(x)^{\alpha\eta-1}(1-x)^{\alpha(1-\eta)-1}\,\Gamma[\alpha]}{\Gamma[\alpha\eta]\,\Gamma[\alpha(1-\eta)]} \qquad x \in [0,1]$$

$$\sum_{k=0}^{n} \mathfrak{p}^{(n)}_k \to n \int_0^1 dx\, \mathfrak{p}^{(n)}_{k=nx} = 1$$

limiting distribution after centering

$$\frac{\mathfrak{p}^n_{nx}}{\mathfrak{p}^n_{n/2}} \sim 2^{\alpha-2} x^{\frac{1}{2}(\alpha-2)}(1-x)^{\frac{\alpha}{2}-1} \quad \to_{x\to y+1/2} \quad \left(1-4y^2\right)^{\frac{1}{2}(\alpha-2)}$$

Wigner law -> q-Gaussian (q=(α-4)/(α-3) and β=2(α-2))

$$\frac{\mathfrak{p}^n_{nx}}{\mathfrak{p}^n_{n/2}}$$

n=1000, 2000
and Wigner

x

$$\langle k \rangle_n = n\eta$$

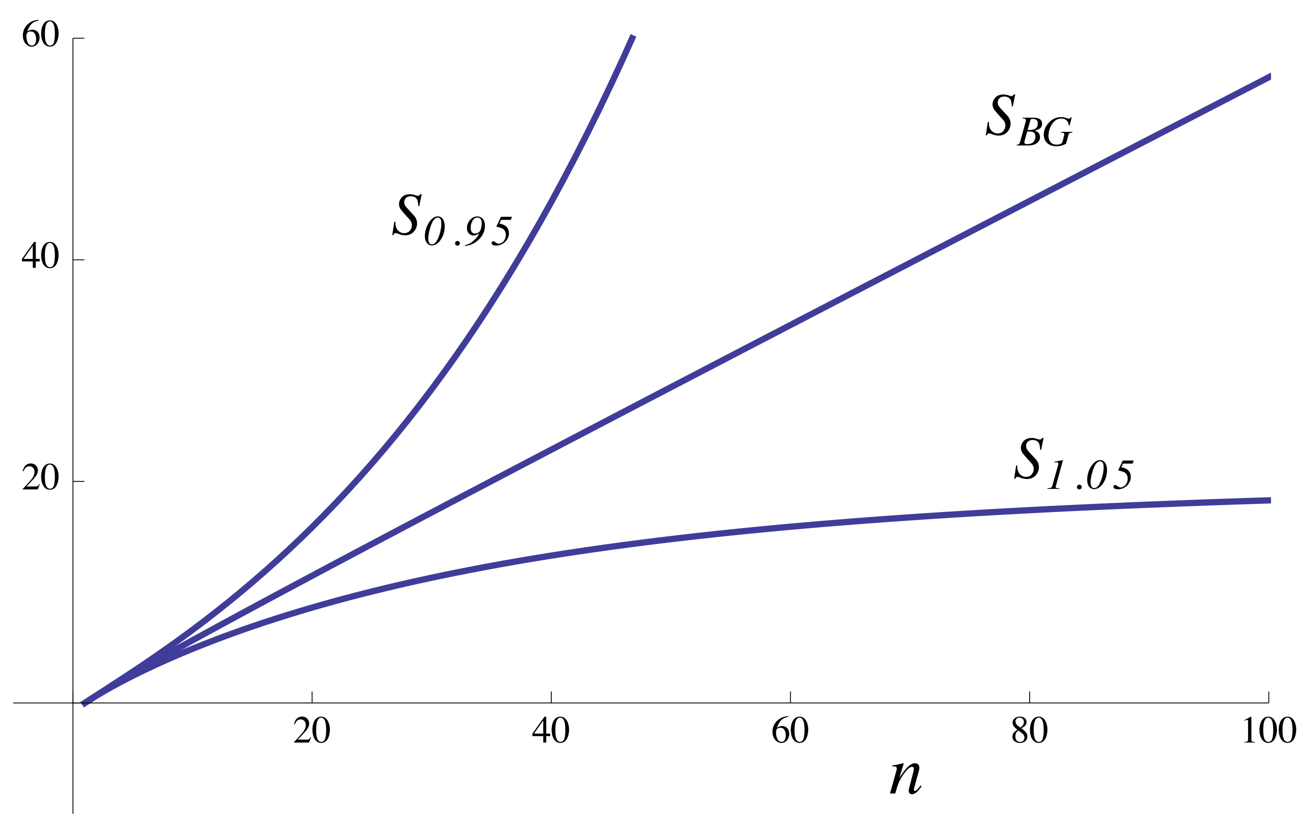
Leibniz triangle rule
is strictly obeyed

$$\sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = \sum_{k=0}^{n} \binom{n}{k} \varpi_k^n = 1$$

$$\varpi_k^{n-1} = \varpi_k^n + \varpi_{k+1}^n$$

Boltzmann-Gibbs and S_q entropies

$$S_{\mathrm{BG}} = -\sum_{k=0}^{n} \binom{n}{k} \frac{\mathfrak{p}_k^{(n)}}{\binom{n}{k}} \log \frac{\mathfrak{p}_k^{(n)}}{\binom{n}{k}} \qquad S_q^{(n)} = \frac{1 - \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\mathfrak{p}_k^{(n)}}{\binom{n}{k}} \right)^q}{q-1}$$



α = 3, η = 1/2

$$S_{BG} \sim n \left[\psi(\alpha) - \eta\psi(\alpha\eta) - (1-\eta)\psi(\alpha(1-\eta)) - \frac{1}{\alpha} \right]$$

$$S_{BG} \sim 0.552961\, n \quad (\alpha = 3;\ \eta = 1/2) \qquad\qquad S_R \sim n \ln 2$$

example 2 - Abel-type polynomials

$$\mathcal{N}(t) = e^{-\alpha W(-t/\alpha)}, \quad \alpha > 0$$

$W(t)e^{W(t)} = t$ W-Lambert's function

$$x_n! = n!\,\frac{\alpha^{n-1}}{(n+\alpha)^{n-1}}$$

$$x_n = \frac{n\alpha}{n+\alpha}\left(1 - \frac{1}{n+\alpha}\right)^{n-2} \qquad \lim_{n\to\infty} x_n = \alpha/e$$

$$q_n(\eta) = \eta\frac{\left(\eta + \frac{n}{\alpha}\right)^{n-1}}{\left(1 + \frac{n}{\alpha}\right)^{n-1}} \qquad q_0(\eta) = 1,\ q_1(\eta) = \eta$$

$$\mathfrak{p}_k^{(n)}(\eta) = \binom{n}{k}\eta(1-\eta)\frac{(\eta + k/\alpha)^{k-1}(1 - \eta + (n-k)/\alpha)^{n-k-1}}{(1 + n/\alpha)^{n-1}}$$

The On-Line Encyclopedia of Integer Sequences®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A063170 Schenker sums with n-th term. 10

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%I
%S 1,2,10,78,824,10970,176112,3309110,71219584,1727242866,46602156800,
%T 1384438376222,44902138752000,1578690429731402,59805147699103744,
%U 2428475127395631750,105224992014096760832,4845866591896268695010
%N Schenker sums with n-th term.
%C Urn, n balls, with replacement: how many selections if we stop after a ball is chosen that
   was chosen already? Expected value is a(n)/n^n.
%C Conjectures: The exponent in the power of 2 in the prime factorization of a(n) (its 2-adic
   valuation) equals 1 if n is odd and equals n - A000120(n) if n is even. - _Gerald McGarvey_,
   Nov 17 2007, Jun 29 2012
%C Amdeberhan, Callan, and Moll (2012) have proved McGarvey's conjectures. - _Jonathan Sondow_,
   Jul 16 2012
%D T. Amdeberhan, D. Callan and V. Moll, Valuations and combinatorics of truncated exponential
   sums, INTEGERS 13 (2013), #A21.
%D D. E. Knuth, The Art of Computer Programming, 3rd ed. 1997, Vol. 1, Addison-Wesley, p. 123,
   Exercise Section 1.2.11.3 18.
%D Helmut Prodinger, An identity conjectured by Lacasse via the tree function, Electronic
   Journal of Combinatorics, 20(3) (2013), #P7
%H T. Amdeberhan, D. Callan, and V. Moll, <a
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asymptotic behavior at large n

$$\mathfrak{P}^{(n)}_{k=nx} \sim \frac{1}{n^{3/2}} \frac{\alpha\,\eta\,(1-\eta)}{\sqrt{2\pi}\,x^{3/2}\,(1-x)^{3/2}}$$

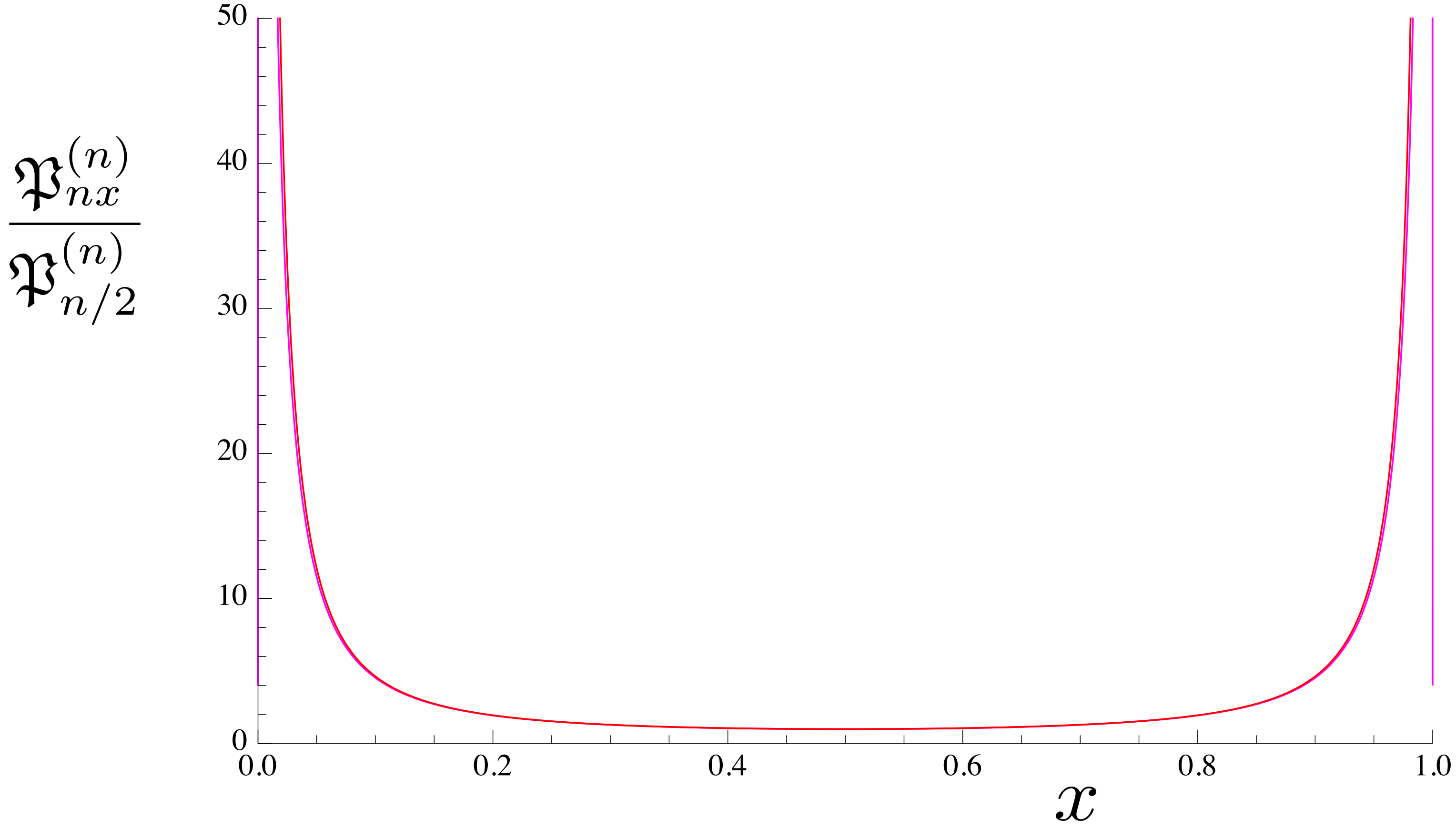
$$\sum_{k=0}^{n} \to n\int_0^1 dx \qquad \sim_{\text{large } n}\ \frac{1}{n^{1/2}} \frac{\alpha\,\eta\,(1-\eta)}{\sqrt{2\pi}} \lim_{\epsilon\to 0}\int_{\epsilon}^{1-\epsilon} \frac{dx}{x^{3/2}\,(1-x)^{3/2}}$$

$$\sim_{\text{large } n}\ \lim_{\epsilon\to 0} \frac{4\alpha\eta(1-\eta)}{\sqrt{2\pi}} \frac{1}{\sqrt{n\epsilon}}$$

$$\epsilon = \frac{A}{n} \qquad A = \frac{8(\alpha\eta(1-\eta))^2}{\pi} \qquad (\text{large } n)\ \ n\int_0^1 dx\,\mathfrak{P}^{(n)}_{k=nx} = 1$$

limiting distribution after centering

$$(\text{large n}) \quad \frac{\mathfrak{P}^{(n)}_{nx}}{\mathfrak{P}^{(n)}_{n/2}} \sim \frac{1}{8[x(1-x)]^{3/2}} \quad \to_{x\to y+1/2} \frac{1}{(1-4y^2)^{3/2}}$$



$n = 20000$

$\alpha = 20$

$\eta = 1/2$

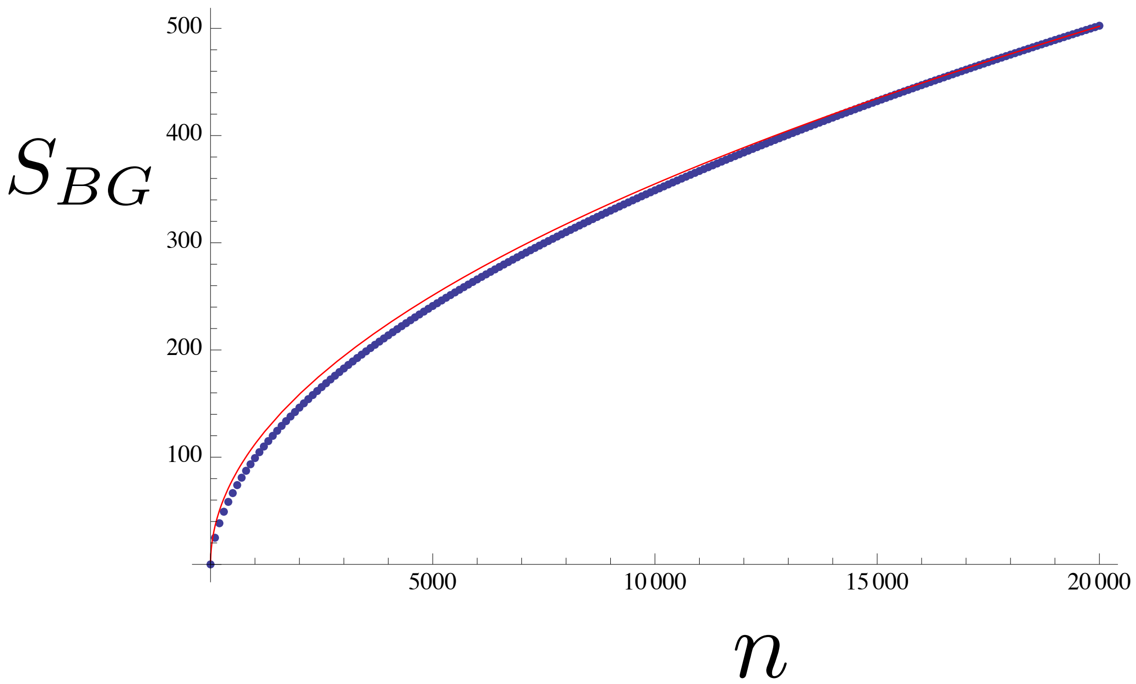
$$\frac{\mathfrak{P}^{(n)}_{nx}}{\mathfrak{P}^{(n)}_{n/2}} \sim \frac{1}{8[x(1-x)]^{3/2}}$$

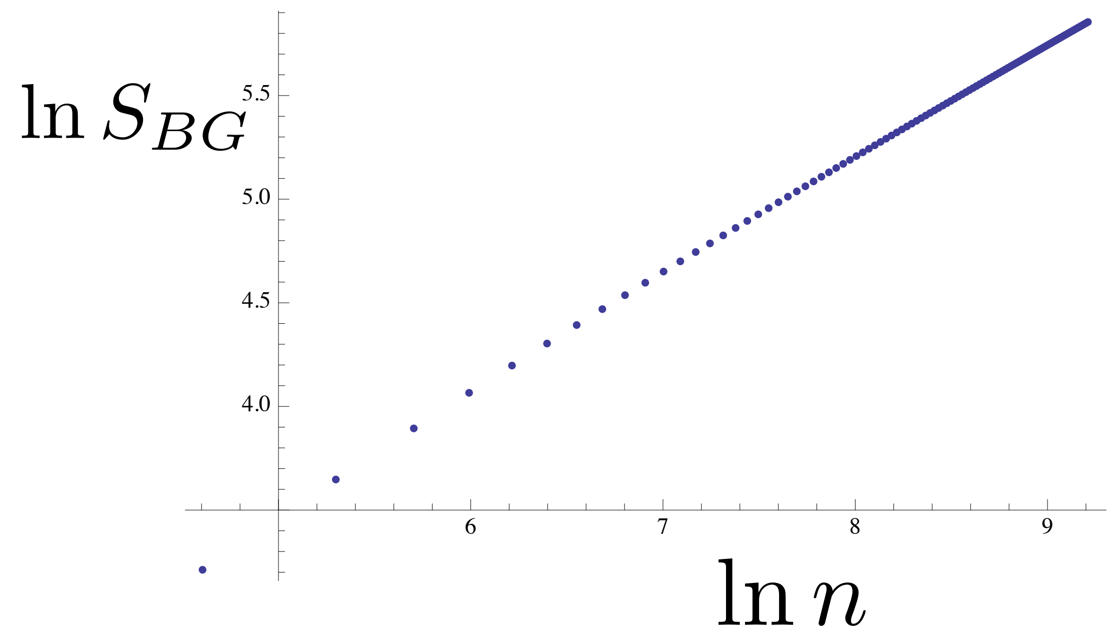
Leibniz triangle rule is asymptotically obeyed, large n

$$\sum_{k=0}^{n} \mathfrak{p}^{(n)}_k(\eta) = \sum_{k=0}^{n} \binom{n}{k} \varpi^n_k = 1$$

(large n) $$\varpi^{n-1}_k \simeq \varpi^n_k + \varpi^n_{k+1}$$

extensive entropies



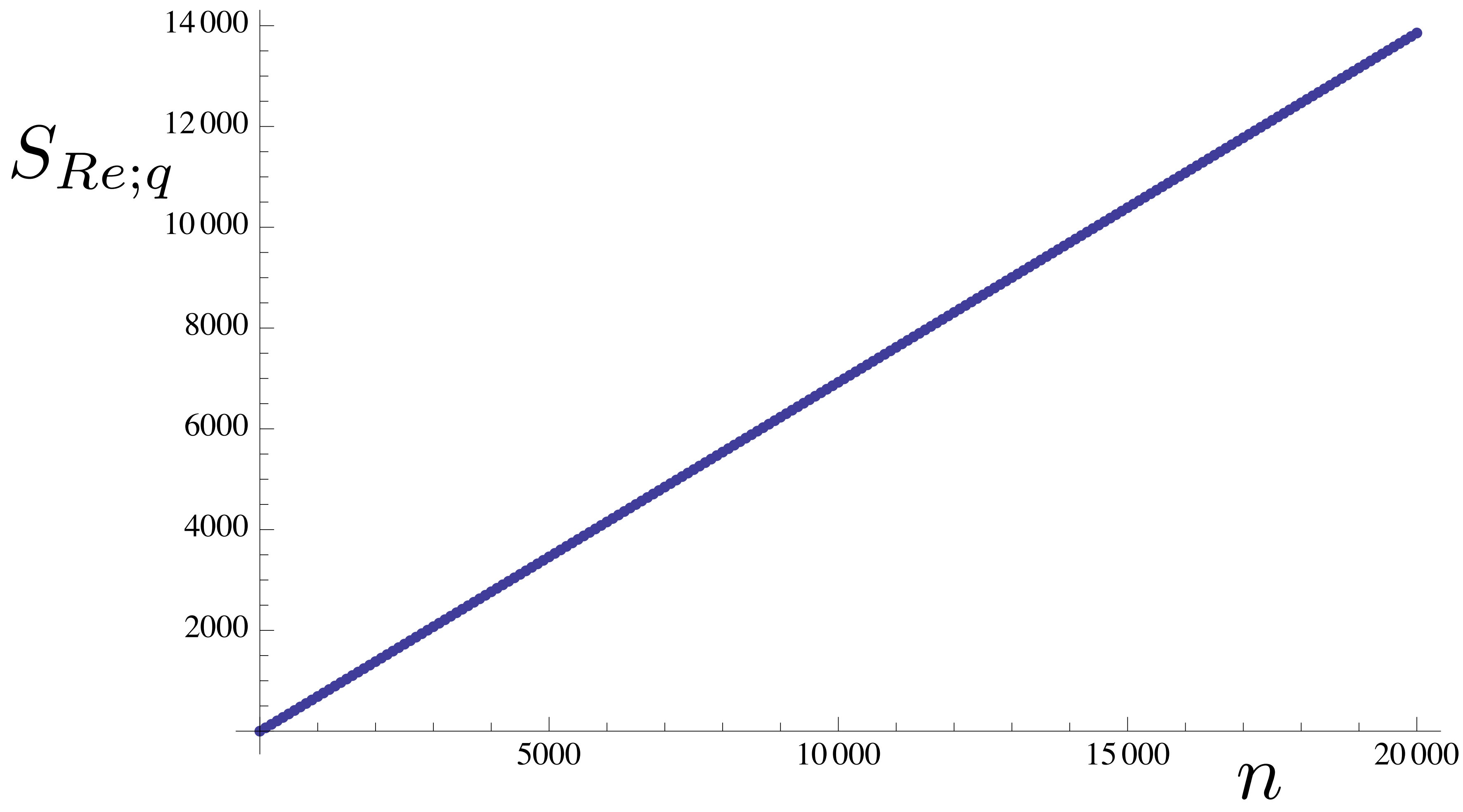


$$(n = 20000;\ \ \alpha = 3;\ \ \eta = 1/2)$$

$$S_{\mathrm{BG}} = -\sum_{k=0}^{n} \binom{n}{k} \frac{\mathfrak{p}_k^{(n)}}{\binom{n}{k}} \log \frac{\mathfrak{p}_k^{(n)}}{\binom{n}{k}} \sim 2\sqrt{2\pi}\, \alpha\, \eta\, (1-\eta)\, \sqrt{n}$$

Rényi entropy

$$S_{\mathrm{Re};q} = \frac{1}{1-q}\log\left[\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\mathfrak{p}_k^{(n)}}{\binom{n}{k}}\right)^q\right] \sim n\log 2$$



$(\alpha = 3,\ \eta = 1/2,\ q = 1/2)$

J Stat Phys (2012) 146:264–280
DOI 10.1007/s10955-011-0383-8

On a Generalization of the Binomial Distribution and Its Poisson-like Limit

E.M.F. Curado · J.P. Gazeau · Ligia M.C.S. Rodrigues

JOURNAL OF MATHEMATICAL PHYSICS **57**, 023301 (2016)

Symmetric deformed binomial distributions: An analytical example where the Boltzmann-Gibbs entropy is not extensive

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Extensivity of Rényi entropy for the Laplace–de Finetti distribution

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asymmetric deformations

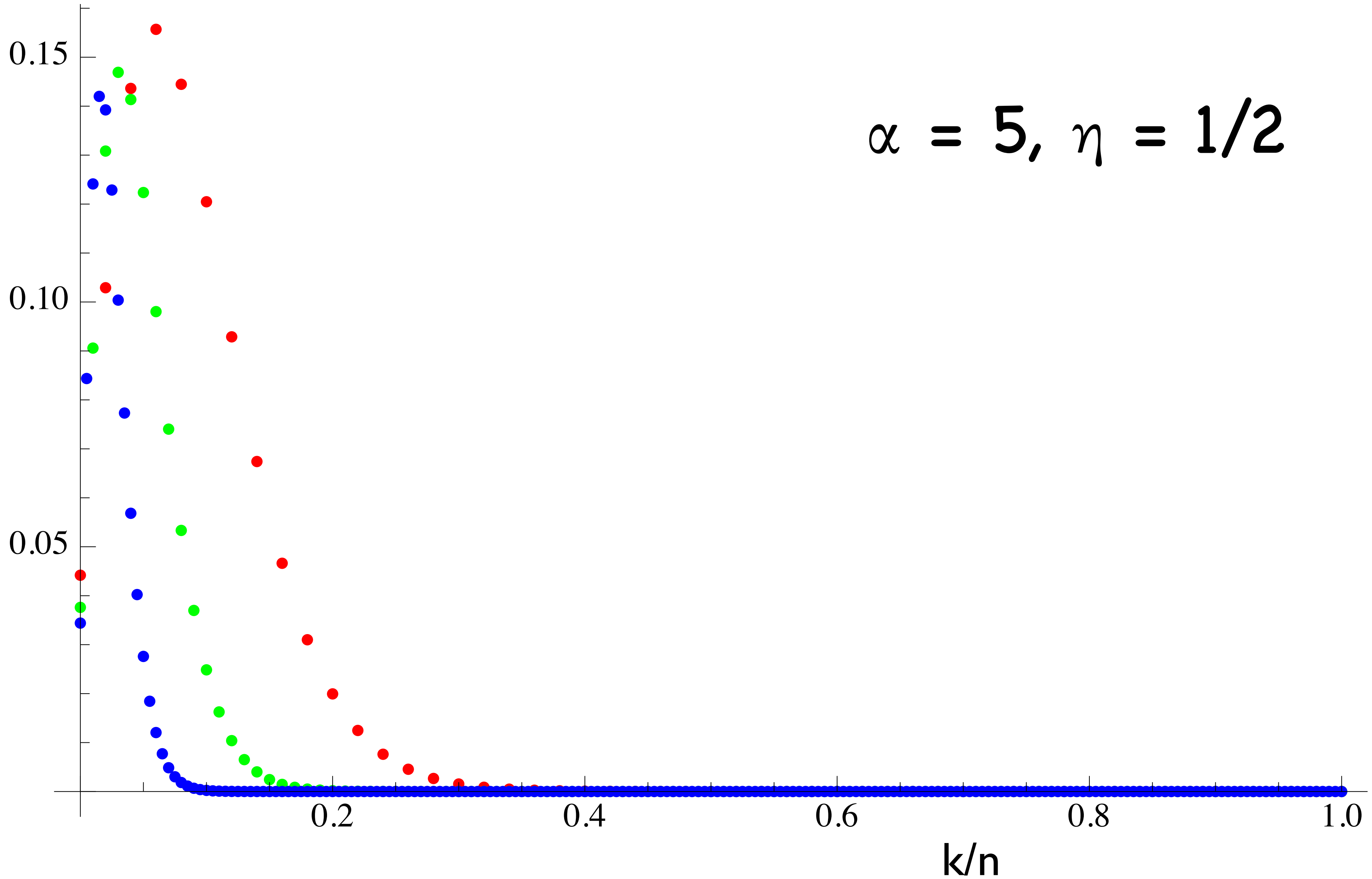
$$\mathfrak{p}_k^{(n)}(\eta) = \frac{x_n!}{x_{n-k}!x_k!}\eta^k p_{n-k}(\eta)$$

deformed probability to have k wins in n trials - regardless the order - induced by correlations

$$\boxed{\mathcal{N}(t) = \left(1 - \frac{t}{\alpha}\right)^{-\alpha}} , \quad \alpha > 0$$

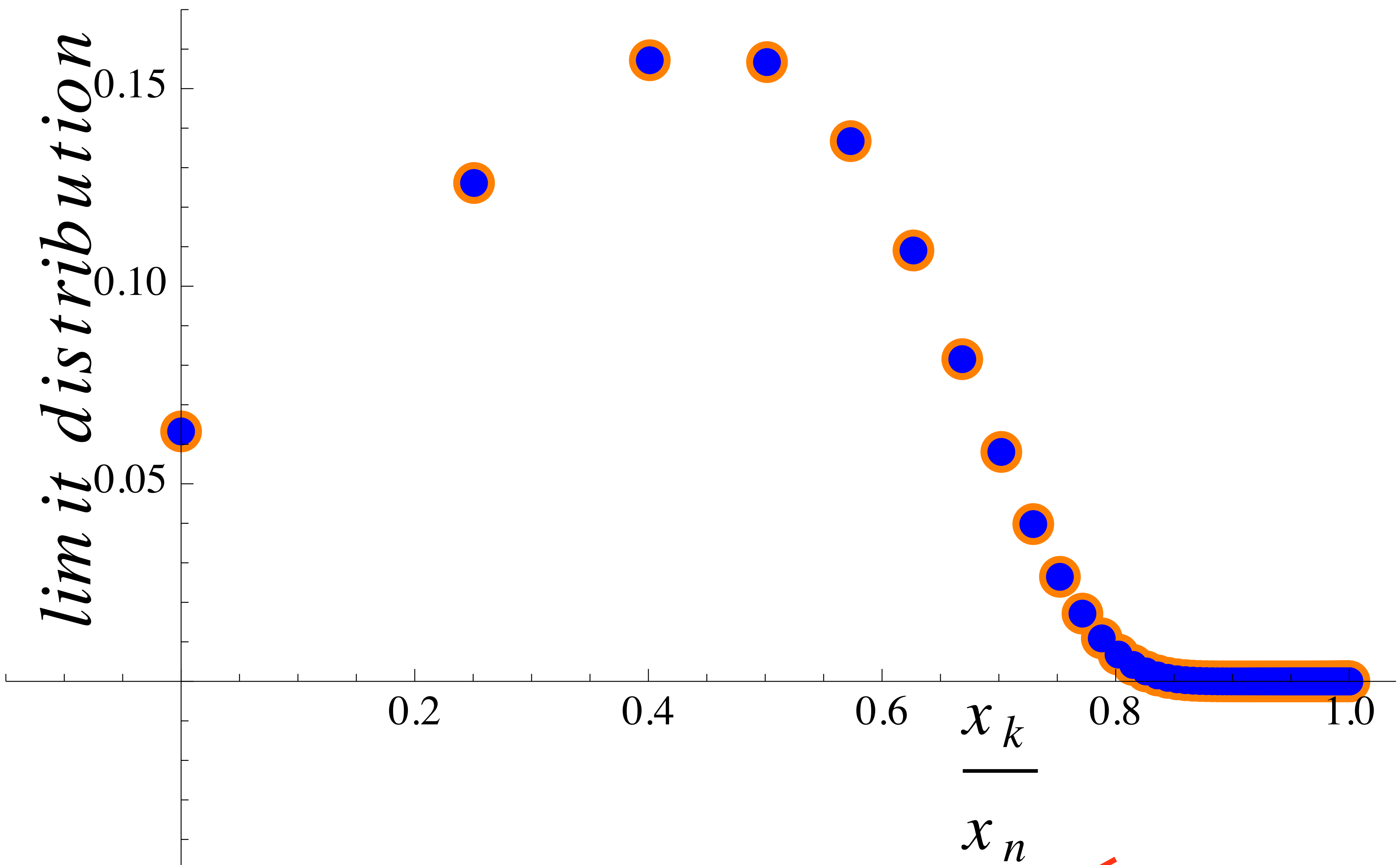
$$S_q = \frac{1}{q-1}\left[1 - \sum_{k=0}^{n}\binom{n}{k}\left(\pi_k^{(n)}\right)^q\right]$$

$$\langle x_k \rangle_n = x_n \eta$$



n = 50, 100, 200

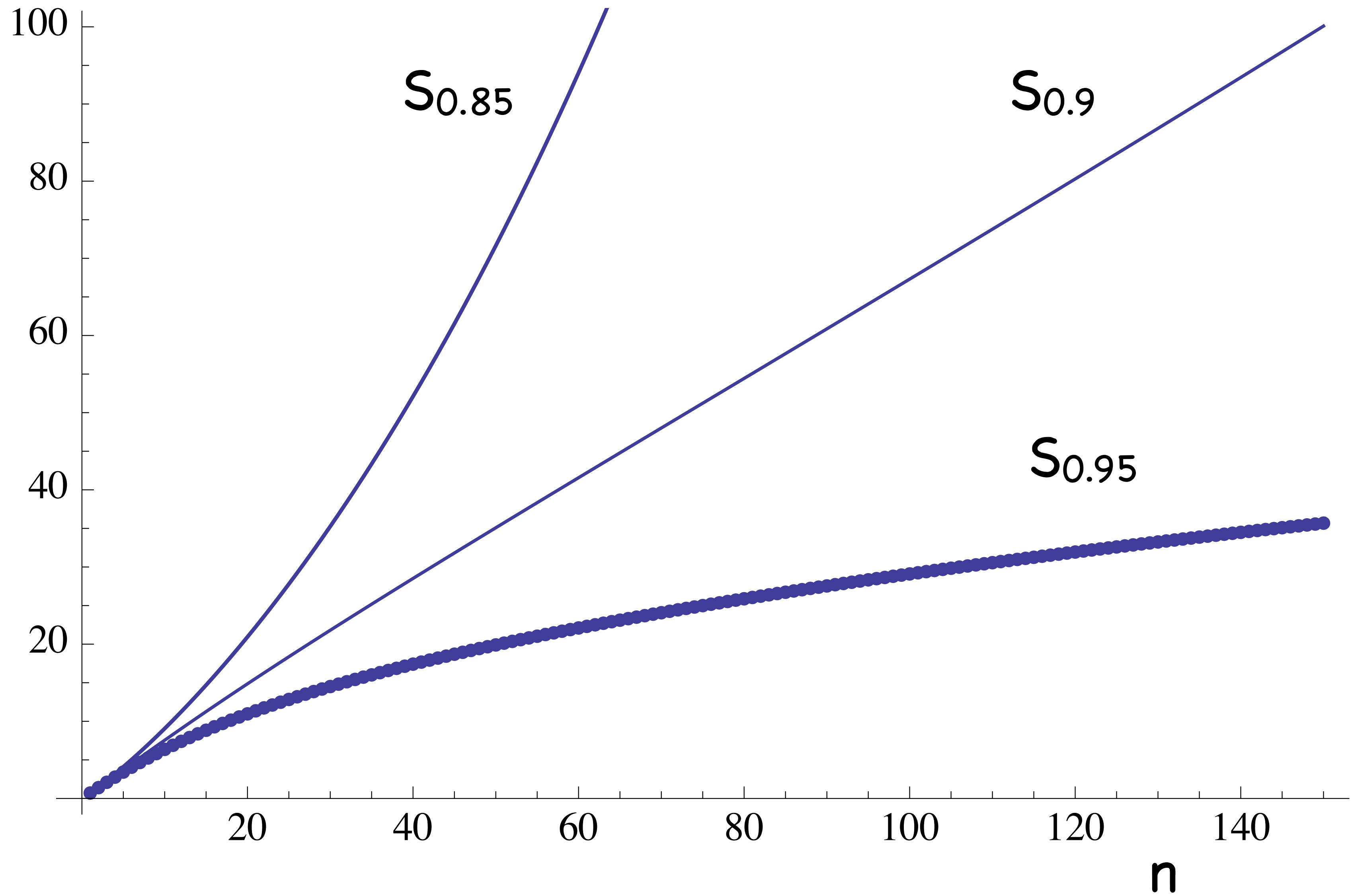
limit distribution



$\langle x_k \rangle_n = x_n \eta$

n = 1000, 2000

a=1, r=4



$q \simeq 0.9$ q < 1! S_q extensive

$$\pi_k^{(n-1)}(\eta) = \pi_k^{(n)}(\eta) + \pi_{k+1}^{(n)}(\eta)$$

Leibniz rule is violated

Journal of Physics A: Mathematical and Theoretical

PAPER

Generalized Heisenberg algebra and (non linear) pseudo-bosons

F Bagarello[1,2], E M F Curado[3] and J P Gazeau[4,5]

Published 15 March 2018 •

Journal of Physics A: Mathematical and Theoretical, Volume 51, Number 15

back to Helstrom bound

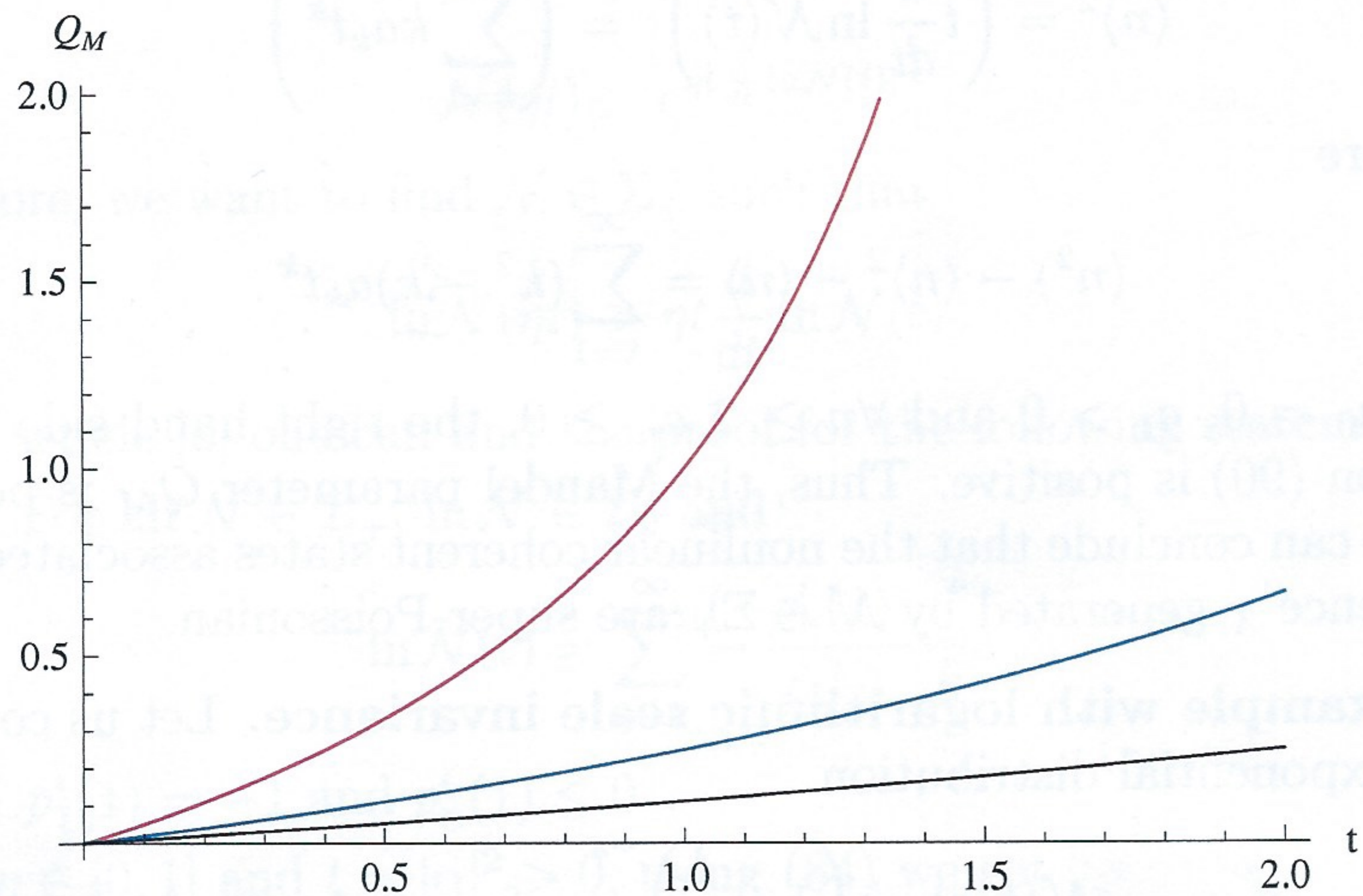
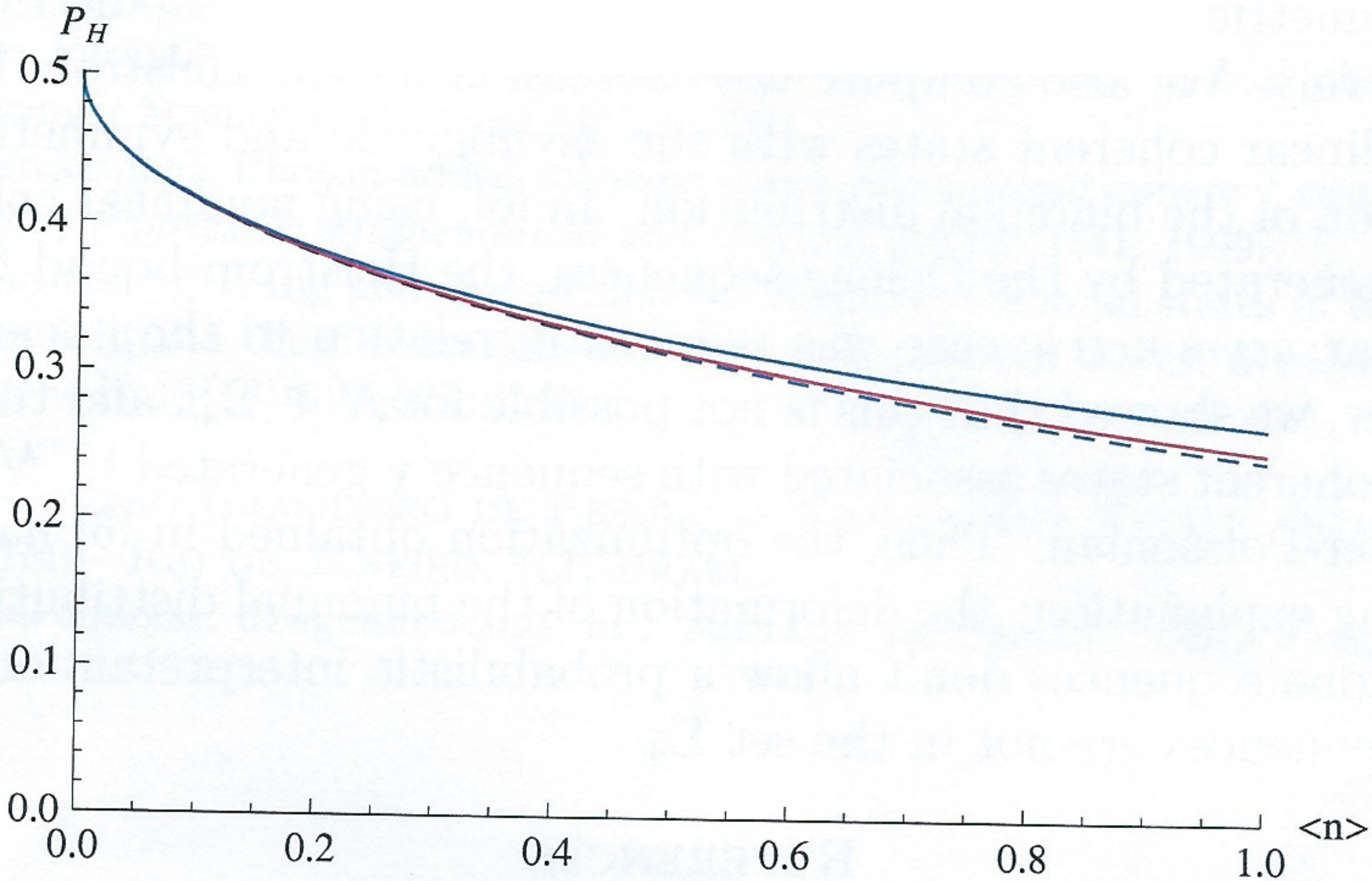


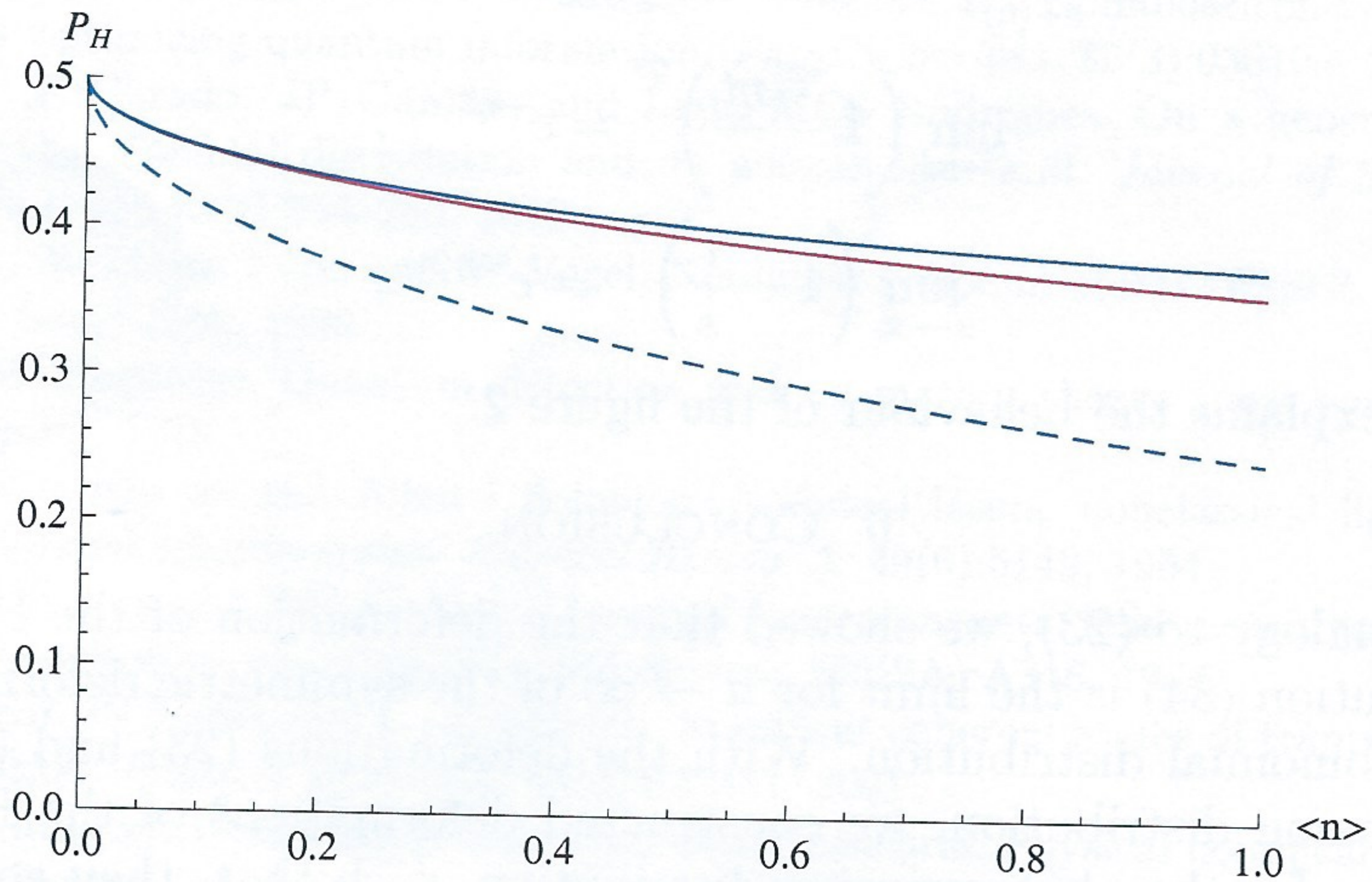
FIGURE 1. The Mandel parameter Q_M for nonlinear coherent states associated with the sequence (92) for $s = 2$ (red line), $s = 5$ (blue line) and $s = 10$ (black line).

$$\mathcal{N}(t,s) = \left(1 - \frac{t}{s}\right)^{-s}$$



(B) Symmetric deformation

blue: s=2; red: s=10



(A) Asymmetric deformation

blue: s=2; red: s=10

obrigado