Computation of critical points for planar statistical physics models

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joint work with V. Beffara / S. Smirnov

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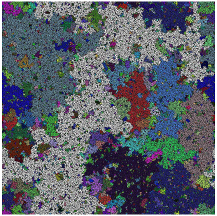
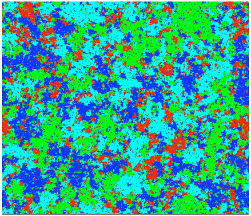
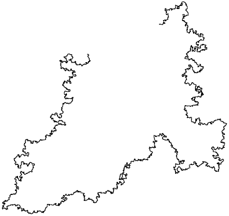

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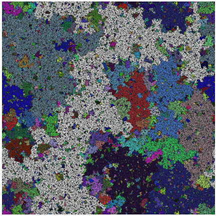
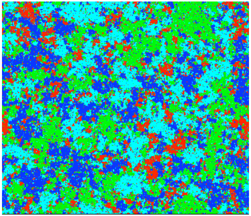
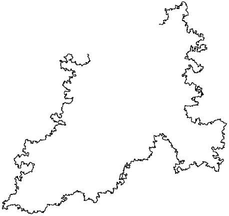

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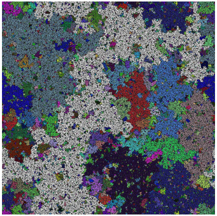
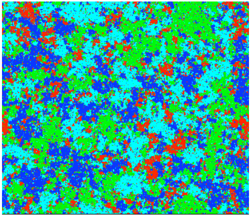
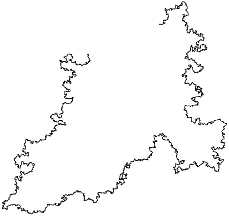
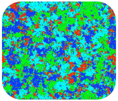






- How do interactions between particles at a microscopic level lead to different behaviors of the model macroscopically?
- We are particularly interested in their phase transition and the behavior at the *critical point*.

TO DO



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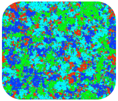

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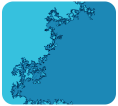

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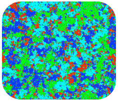

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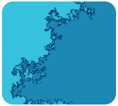

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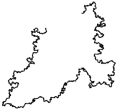

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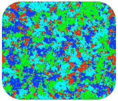

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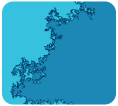

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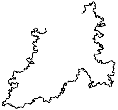

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This model undergoes a phase transition at some $p_c \in (0,1)$:

- when $p < p_c$, $\phi_p(0 \longleftrightarrow \infty) = 0$,
- when $p > p_c$, $\phi_p(0 \longleftrightarrow \infty) > 0$.

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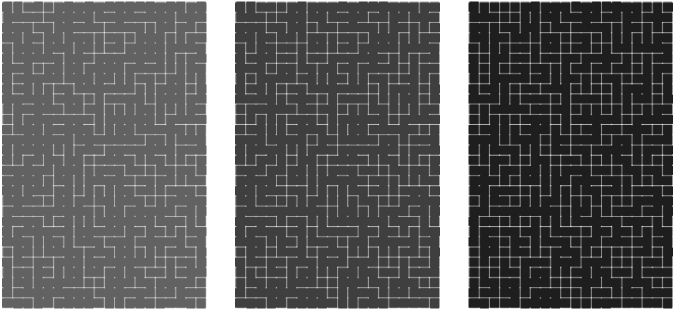

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Remains to prove that $p_c > 0$ and $p_c < 1$ (Peirls argument which is **combinatorial in nature**).

Harder case: Ising model

Assign to each site outside $[-n,n]^2$ the spin $+1$ and each site of $[-n,n]^2$ a spin $+1$ or -1 according to the following probability measure:

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- For $T > T_c$, $\mathbb{P}^+_{T,2,n}[\sigma_0 = +]$ tends to $\frac{1}{2}$ as $n \to \infty$
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Assign to each site outside $[-n, n]^2$ the color red and each site of $[-n, n]^2$ a color amongst q colors according to the following probability measure:

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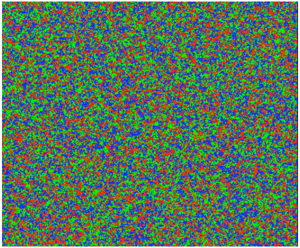

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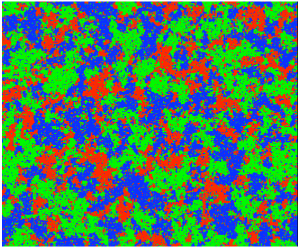

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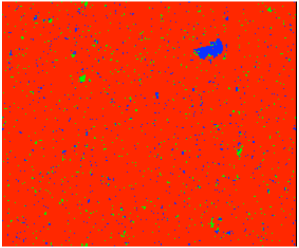

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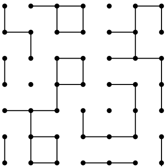

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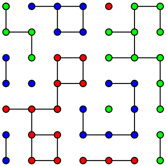

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It is sufficient to compute $p_c(q)$ for FK percolation with $q \geq 1$

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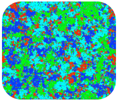

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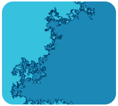

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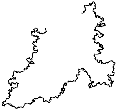

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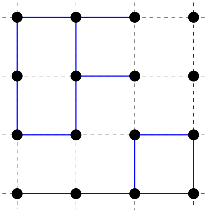

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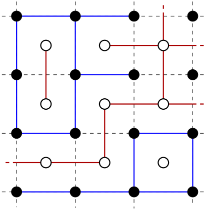

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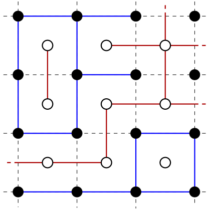

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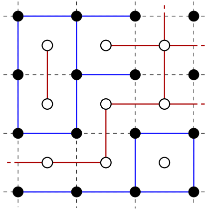

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If $p^\star = p$, i.e. $p = p_{sd} = \sqrt{q}/(1+\sqrt{q})$, the primal and dual models play symmetric roles.

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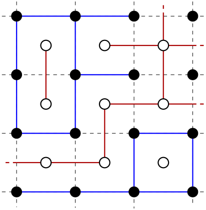

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- For FK percolation, it is a FK percolation with $p^\star$ and $q^\star$ defined by

$$q^\star = q \quad \text{and} \quad \frac{pp^\star}{(1-p)(1-p^\star)} = q.$$

If $p^\star = p$, i.e. $p = p_{sd} = \sqrt{q}/(1+\sqrt{q})$, the primal and dual models play symmetric roles. For instance, $\phi_{p_{sd},q,n}(A_n) = \frac{1}{2}$.

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Proposition (Margulis/Russo's formula)

For any increasing event A,

$$\phi_{p+\varepsilon}[A] - \phi_p[A] = \mathbb{P}[\omega_{p+\varepsilon} \in A \text{ and } \omega_p \notin A]$$
$$= \Big(\sum_{e\in E} \phi_p(\omega^e \in A \text{ and } \omega_e \notin A)\Big)\varepsilon + o(\varepsilon).$$

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At least one site must have a 'large' influence when the event has probability away from 0 and 1.

Theorem (Kahn, Kalai & Linial 1988 – Bourgain, K., K., Katznelson & L. 1992)

For every increasing event A on the graph $[-n,n]^2$,

$$\max_{e\in E} \phi_p(\omega^e \in A \text{ and } \omega_e \notin A) \geq c\, \phi_p[A]\big(1-\phi_p[A]\big)\frac{\log n}{n^2}.$$

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The difficulty lies mostly in this last step!

But it applies to $q > 1$ as well!

Theorem (Beffara, D-C, 2010)

The critical point $p_c(q)$ of the FK percolation on the square lattice satisfies

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

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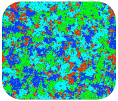
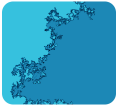
Conclusion: This general philosophy has been very successful thanks to its robustness. Ongoing works suggest that this approach can be implemented for a wide class of models, known as **positively correlated models**, which are natural candidates for geometric representations of spin models.

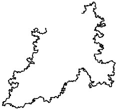
TO DO



I. Why is there a phase transition?

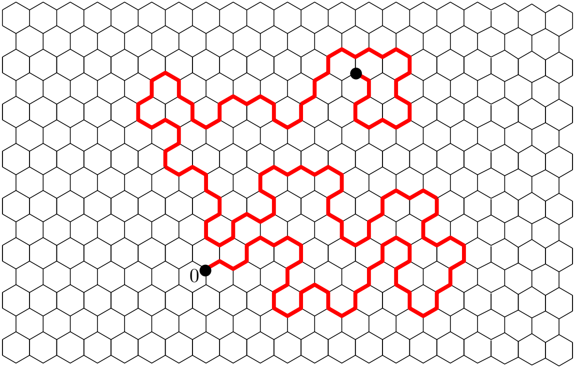


II. Computation of the critical point via self-duality (the example of the FK percolation)



III. Computation of the critical point via discrete integrability (the example of the SAW)

On a lattice (for instance hexagonal $\mathbb{H}$), consider self-avoiding trajectories (or walks) of length n (the length is denoted by $\ell(\gamma)$) starting at the origin. Introduced by Flory and Ott in the '50s.



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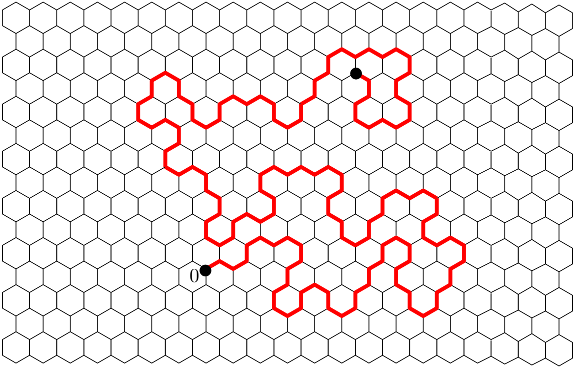

- **Combinatorial question:** What is the **asymptotic behavior** of the number of self-avoiding walks of length n?

Proposition

Let $c_n := \#$ SAW of length n. Then, $c_n = \mu_c^{n+o(n)}$, where μ_c is the connective constant.

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➡ The *foundamental subadditive lemma* of Fekete implies the result. ☺

The connective constant μ_c as a critical parameter?

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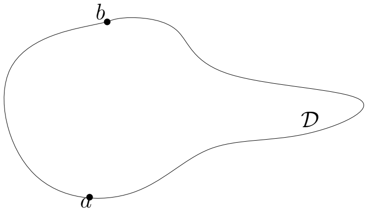

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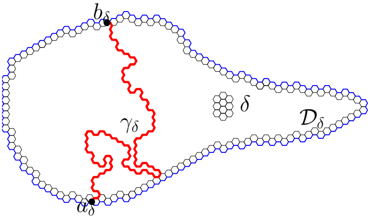

- For $\delta > 0$, we define a probability measure on self-avoiding paths from a_δ to b_δ by assigning a weight proportional to $\mu^{-\ell(\gamma)}$. When $\delta \to 0$, we are interested in the limit of this sequence of **random continuous curves** (scaling limit).

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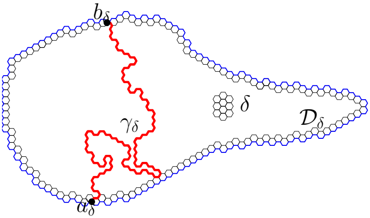

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Theorem (Ioffe, 1998)

For $\mu > \mu_c$, the scaling limit of the SAW is a line.

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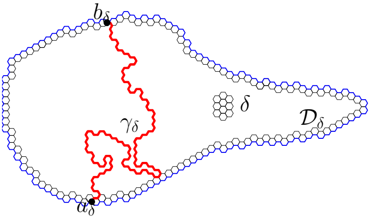

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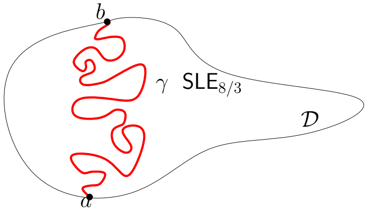
Theorem (D.-C., Kozma, Yadin, 2012)

For $\mu < \mu_c$, the scaling limit of the SAW is space filling.

The connective constant μ_c as a critical parameter?



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Conjecture (Lawler, Schramm, Werner, 2001)

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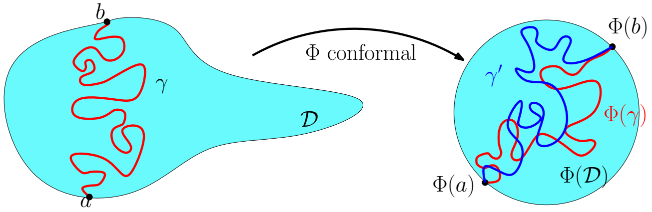

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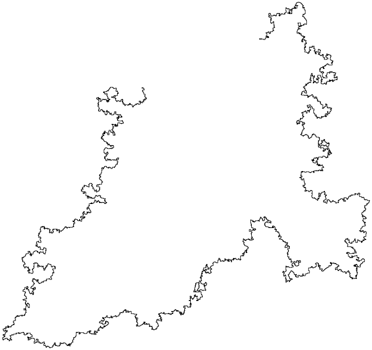
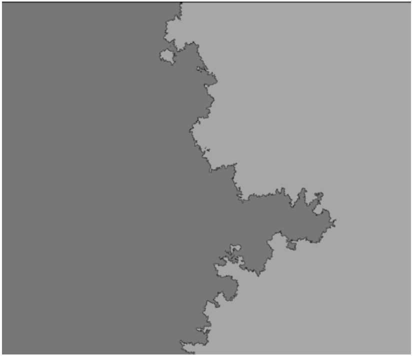
Conjecture (Lawler, Schramm, Werner, 2001)

For $\mu = \mu_c$, the scaling limit of the SAW is SLE(8/3) which is **conformally invariant**.

1000 steps Self-avoiding walk and SLE(8/3)





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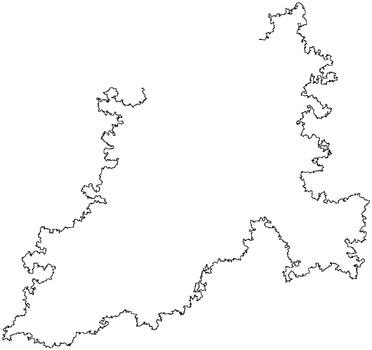

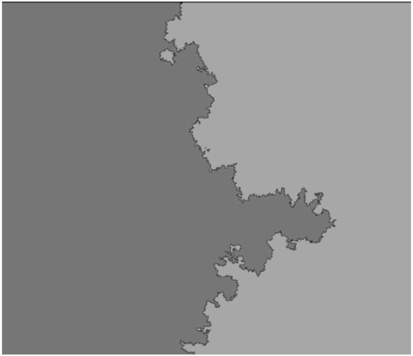

Theorem (D-C, Smirnov, 2010),

The connective constant μ_c of the **hexagonal lattice** satisfies

$$\mu_c := \lim_{n\to\infty} c_n^{\frac{1}{n}} = \sqrt{2+\sqrt{2}}.$$

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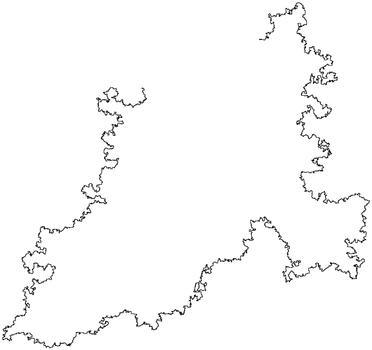
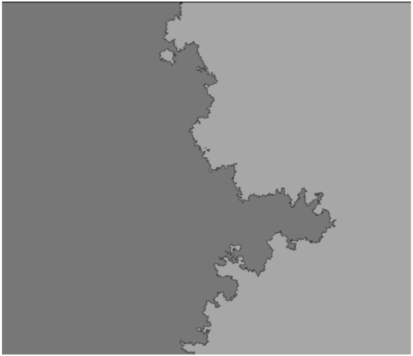


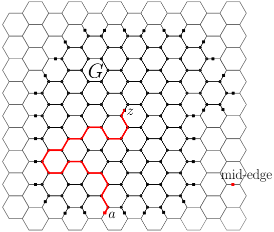


Theorem (D-C, Smirnov, 2010), conjectured by Nienhuis (1980)

The connective constant μ_c of the **hexagonal lattice** satisfies

$$c_n \sim An^{11/32}\sqrt{2+\sqrt{2}}^{\,n} \text{ as } n \longrightarrow \infty$$

We restrict our attention to *finite domains* $\mathcal{D}$ and we weight walks by $\mu^{-\ell(\gamma)}$ times a topological term depending on the *winding*.



$$F(z) := \sum_{\gamma \subset \mathcal{D}:\ a \to z} \mu^{-\ell(\gamma)}.$$

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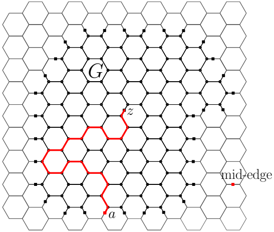

Definition

The winding $\mathrm{W}_\Gamma(a, b)$ of a curve Γ between a and b is the rotation (in radians) of the curve between a and b.

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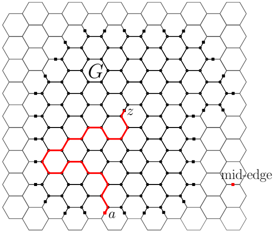

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The **parafermionic operator** at a mid-point $z \in \mathcal{D}$ is defined by

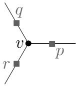
$$F(z) := \sum_{\gamma \subset \mathcal{D}:\ a \to z} \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} \mu^{-\ell(\gamma)}.$$

Lemma (Local relation around a vertex)

If $\sigma = \frac{5}{8}$ and $\mu = \sqrt{2+\sqrt{2}}$, then F satisfies the following relation for every vertex $v \in V(\mathcal{D})$,

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0$$

where p, q, r are the mid-edges of the three edges adjacent to v.



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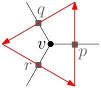
This relation means that $\oint F(z)dz = 0$ along the contour



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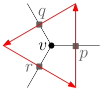

Proposition (Discrete holomorphicity)

If $\mathcal{D}$ is simply connected, then $\oint_\gamma F(z)dz = 0$ **for any** *discrete contour γ.*

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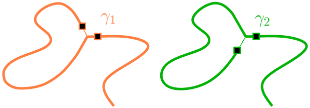
One can partition the set of walks γ finishing at p, q or r into **pairs** and **triplets** of walks:



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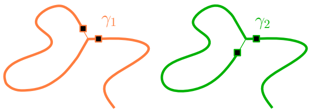

In the first case,

$$c(\gamma_1) + c(\gamma_2) = (q - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,q)}\mu^{-\ell(\gamma_1)} + (r - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_2}(a,r)}\mu^{-\ell(\gamma_2)}$$
$$= (p - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)}\mu^{-\ell(\gamma_1)}\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\sigma\cdot\frac{-4\pi}{3}} + \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\sigma\cdot\frac{4\pi}{3}}\right)$$

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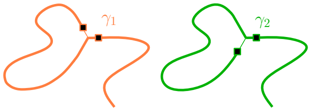

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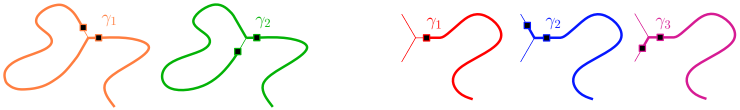

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In the second case,

$$c(\gamma_1) + c(\gamma_2) + c(\gamma_3)$$

$$= (p - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)}\mu^{-\ell(\gamma_1)}\left(1 + \mu^{-1}\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{-\pi}{3}} + \mu^{-1}\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{\pi}{3}}\right).$$

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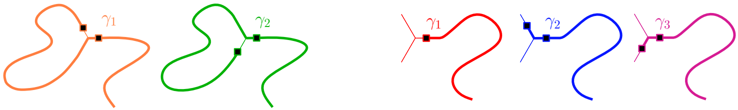

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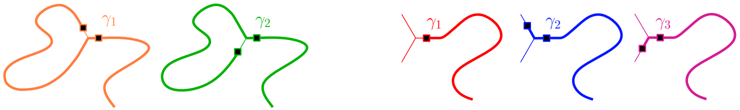
$$= (p - v)\mathrm{e}^{-\mathrm{i}\frac{5}{8}\mathrm{W}_{\gamma_1}(a,p)}\mu^{-\ell(\gamma_1)}\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{-4\pi}{3}} + \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{4\pi}{3}}\right) = 0$$

In the second case, providing $\mu := \sqrt{2+\sqrt{2}}$,

$$c(\gamma_1) + c(\gamma_2) + c(\gamma_3)$$
$$= (p - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)}\mu^{-\ell(\gamma_1)}\left(1 + \mu^{-1}\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{-\pi}{3}} + \mu^{-1}\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\frac{5}{8}\cdot\frac{\pi}{3}}\right) = 0.$$

We write $c(\gamma)$ for the contribution of the walk γ to the sum.

One can partition the set of walks γ finishing at p, q or r into **pairs** and **triplets** of walks:



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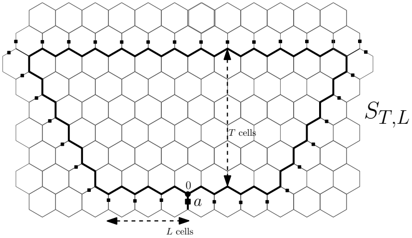
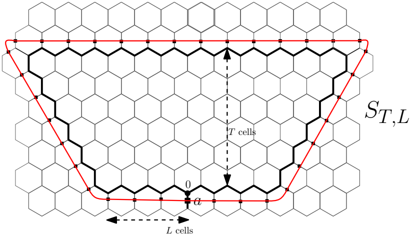
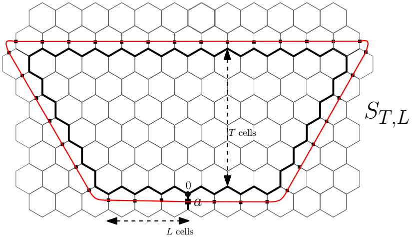

$S_{T,L}$
T cells
0
a
L cells



If we consider the exterior boundary of the domain, we obtain

When $\sigma = \frac{5}{8}$ and $\mu = \sqrt{2+\sqrt{2}}$, we have

$$0 = -\sum_{z\in bottom} F(z) + \sum_{z\in top} F(z) + e^{i\frac{2\pi}{3}} \sum_{z\in left} F(z) + e^{-i\frac{2\pi}{3}} \sum_{z\in right} F(z)$$



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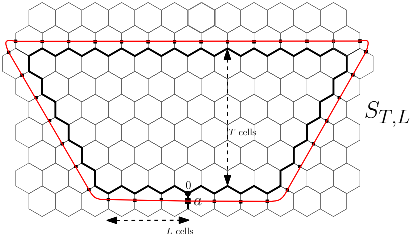
We know the winding on the boundary! Thus, we can replace F by the sum of Boltzman weights.



If we consider the exterior boundary of the domain, we obtain

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$$1 = \cos\left(\frac{3\pi}{8}\right) \sum_{\gamma:a\to \text{bottom}} \mu^{-\ell(\gamma)} + \sum_{\gamma:a\to \text{top}} \mu^{-\ell(\gamma)} + \cos\left(\frac{\pi}{4}\right) \sum_{\gamma:a\to \text{sides}} \mu^{-\ell(\gamma)}.$$

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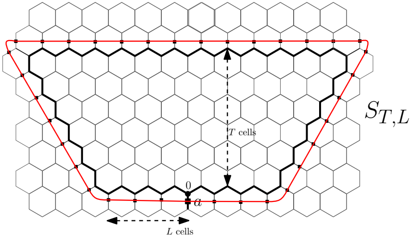

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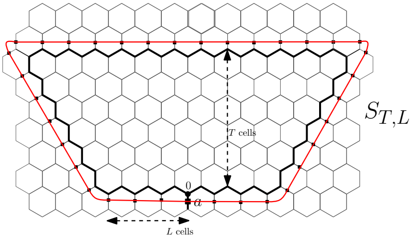

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- The result follows from this combinatorial relation.

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Therefore, the number of walks $b_{n,T}$ of length n and height T never going below their start and above their end satisfies $b_{n,T} \leq \mu^n$.

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The number of such walks is the same (at the exponential scale) as the number of unconstrained walks (use an unfolding argument). Therefore,

$$\mu_c(\mathbb{H})^{n+o(n)} = b_n \le n\mu^n.$$

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- The method by **sharp threshold** is more general and applies to a wide variety of models.

Thank you