Dynamically Consistent Objective and Subjective Rationality

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Introduction

- Consider a group of experts acting as a decision maker (DM) choosing among two policies, f and g.
    - For instance a group of epidemiologists advising a prime minister about the best policy to fight the Coronavirus outbreak in 2020: f is lockdown the population and g is herd immunity.

Introduction (cont'd)

- Assume that each member is represented by a Subjective Expected Utility preference, they share the same attitude toward risk but differ on their probability distribution.
  - Expert 1 has a probability P_1 over states of nature (e.g. the virus is severe, mild or innocuous).

$$f \succsim_1 g \Leftrightarrow \mathbb{E}_{P_1}[u(f)] \geq \mathbb{E}_{P_1}[u(g)]$$

  - Let $\mathcal{C}$ be the set of all probabilities (the set of experts).

Objective and Subjective Rationality

- If everybody agree that f is better than g, i.e. **unanimity** holds:

$$f \succsim g \Leftrightarrow \mathbb{E}_P[u(f)] \geq \mathbb{E}_P[u(g)], \text{ for all } P \in \mathcal{C}$$

- However, this is not always the case and decisions must be made nonetheless.
    - ∗ All experts in the group have a veto power: it is sufficient that one expert ranks g above f to break unanimity.

Precautionary Principle

- In this situation, especially when uncertainty about different scenarios is high and there are scenarios that can lead to catastrophes, several authors suggest to adopt the **precautionary principle**.
  - ∗ While there is not an accepted and universal definition, one can think of it as saying that a policy should be evaluated through the opinion of the most pessimistic expert.

Precautionary Principle (cont'd)

- Gilboa, Maccheroni, Marinacci and Schmeidler (2010, GMMS) offer an axiomatic foundation supporting the use of the Maxmin Expected Utility (MEU) of Gilboa and Schmeidler (1989, GS) in order to "complete" the unanimity rule.
    - **Default to Certainty** favors certainty if there is lack of unanimity based on hard evidence and it behaviorally justifies the identification of the MEU rule with the precautionary principle.

Objective and Subjective Rationality

- We follow GMMS approach of a DM characterized by two binary relations capturing two notions of rationality.
  - ∗ The **objective rationality**, denoted by $\succsim^*$, captures decisions supported by the unanimity rule over beliefs à la Bewley (2002):

$$f \succsim^* g \Leftrightarrow \mathbb{E}_P[u(f)] \geq \mathbb{E}_P[u(g)], \text{ for all } P \in \mathcal{C}$$

  - ∗ The **subjective rationality**, denoted by $\succsim^{\#}$, captures situations where the precautionary principle applies à la GS:

$$f \succsim^{\#} g \Leftrightarrow \min_{P \in \mathcal{C}} \mathbb{E}_P[u(f)] \geq \min_{P \in \mathcal{C}} \mathbb{E}_P[u(g)]$$

Theorem GMMS

Objective and Subjective Rationality (cont'd)

Example 1

Consider an urn containing 90 balls, 30 of which are red (R), while the remaining 60 are either blue (B) or green (G) with the number of blue balls being between 15 and 45. A group of experts, with the same attitude toward risk can be identified with the set

$$\mathcal{C} = \left\{ P = \left(\frac{1}{3}, p, \frac{2}{3} - p \right) \in \Delta \middle| p \in \left[\frac{1}{6}, \frac{1}{2} \right] \right\}.$$

Suppose they have to choose between the two acts:

	Red	Blue	Green
f	10	0	10
g	0	10	10

Objective and Subjective Rationality (cont'd)

Example 1 (cont'd)

Acts f and g are not comparable w.r.t. $\succsim^*$. For instance let $P_1 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$, $P_2(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$ and suppose w.l.g. $u(0) = 0$:

$$\mathbb{E}_{P_1}[u(f)] = \frac{5}{6}u(10) > \frac{4}{6}u(10)\mathbb{E}_{P_1}[u(g)]$$
$$\mathbb{E}_{P_2}[u(f)] = \frac{1}{2}u(10) < \frac{2}{3}u(10)\mathbb{E}_{P_2}[u(g)]$$

Applying the precautionary principle, we have

$$\min_{P \in \mathcal{C}} \mathbb{E}_P[u(f)] = \frac{1}{2}u(10) < \frac{2}{3}u(10) = \min_{P \in \mathcal{C}} \mathbb{E}_P[u(g)],$$

i.e. $g \succ^{\#} f$.

Dynamic Consistency Problems

- However, if we introduce an intermediary period of partial resolution of uncertainty, the DM may feel embarrassed because of violation of **Dynamic Consistency** on the subjectively rational preference $\succsim^{\#}$.
  * That is, choices made today are regretted tomorrow, no matter the piece of information learned.

Main contribution

- We provide a **refinement** of the concept of objective and subjective rationality that precludes violations of Dynamic Consistency.
    - ∗ The main idea is that the group of experts should take into account today the structure of information that will be partially revealed tomorrow.

Main contribution (cont'd)

- We propose two axioms, **Ex-Ante** and **Ex-Post Coherence**, that allow us to derive the **coherent precautionary reassessment** of the objective rationality.
    * Given that, the initial group of experts is modified by "adding" new experts until one obtains the smallest "rectangular set" that contains the original one.
- By product, we provide a novel behavioral characterization of **rectangularity** and a prescriptive way to aggregate opinions to avoid sure regret.
    * As showed by Sarin and Wakker (1998) and Epstein and Schneider (2003), a rectangular set of priors is a necessary and sufficient condition for Dynamic Consistency to be satisfied by a MEU preference.

Framework

- We consider a set S of states of nature endowed with a sigma-algebra of events Σ. The DM doesn't know which $s \in S$ will occur.
- She has preferences over random variables called acts, $f : S \to X$ (Σ-measurable functions), where X is a convex subset of a vector space (e.g., set of finite support monetary lotteries). $\mathcal{F}$ denotes the set of all acts.
- Given an event E (a subset of S) and two acts f and g, we define the act $fEg \in \mathcal{F}$ by

$$(fEg)(s) = \begin{cases} f(s), & s \in E \\ g(s), & s \in E^c \end{cases}$$

Framework

- We also assume an intermediate period of **partial resolution of uncertainty** which is described by a finite partition $\mathcal{P} = \{E_1, \cdots, E_n\}$.
  - * The DM knows today that tomorrow she will learn that $s \in E_i$ for some $i = 1, \cdots, n$.
- We call $\succsim$ the unconditional or ex-ante preference and $\succsim_E$ denotes the conditional or ex-post preference given E. The ex-ante preference $\succsim$ will be associated to a set of priors $P \in \mathcal{C}$ and we will assume that $P(E_i) > 0$ for all $i \in \{1, \cdots, n\}$ and all $P \in \mathcal{C}$.
- $\mathcal{C}^E$ denotes the **full Bayesian update** of $\mathcal{C} \subset \Delta$:

$$\mathcal{C}^E = \{P^E | P \in \mathcal{C}\} \text{ where } P^E(\cdot) = \frac{P(\cdot \cap E)}{P(E)}.$$

Objective and Subjective Rationality

- Remember that preference $\succsim^*$ is the "unanimity rule" of Bewley (2002):

$$f \succsim^* g \Leftrightarrow \mathbb{E}_P[u(f)] \geq \mathbb{E}_P[u(g)], \text{ for all } P \in \mathcal{C}$$

and $\succsim^\#$ is its "completion" with the precautionary principle (Default to Certainty), represented by a MEU preference of Gilboa and Schmeidler (1989):

$$f \succsim^\# g \Leftrightarrow \min_{P \in \mathcal{C}} \mathbb{E}_P[u(f)] \geq \min_{P \in \mathcal{C}} \mathbb{E}_P[u(g)]$$

Dynamic (in)Consistency (cont'd)

- **Dynamic consistency**: For all $f, g \in \mathcal{F}$ and $E \in \mathcal{P}$,

$$f \succsim_E g \Leftrightarrow fEg \succsim g.$$

- Given DC, it is well-known that $\succsim_E^*$ is also a Bewley preference that can be represented by $(u, \mathcal{C}^E)$.
- Moreover, the pair $(\succsim^*, \succsim_E^*)$ always satisfies Dynamic Consistent.
- Note that, in most of the cases, if $\succsim^*$ is incomplete then $\succsim_E^*$ is also incomplete.
  * Using the precautionary principle to complete $\succsim_E^*$ with a MEU preference $\succsim_E^\#$, dynamic inconsistencies may arise between $\succsim^\#$ and $\succsim_E^\#$

Dynamic (in)Consistency (cont'd)

Example 1 (cont'd)

Consider an urn containing 90 balls, 30 of which are red (R), while the remaining 60 are either blue (B) or green (G) with the number of blue balls being between 15 and 45. A group of experts, with the same attitude toward risk can be identified with the set

$$\mathcal{C} = \left\{ P = \left(\frac{1}{3}, p, \frac{2}{3} - p \right) \in \Delta \middle| p \in \left[\frac{1}{6}, \frac{1}{2} \right] \right\}.$$

Suppose they have to choose between the two acts:

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Dynamic (in)Consistency (cont'd)

Example 1 (cont'd)

Lets add an intermediary period of partial resolution of uncertainty,

$$\mathcal{P} = \{G, RB\}.$$

Suppose the experts update the probabilities in $\mathcal{C}$ with the full Bayesian rule, that is,

$$\mathcal{C}^G = \{(0,0,1)\} \text{ and } \mathcal{C}^{RB} = \left\{(q, 1-q, 0) \in \Delta \middle| q \in \left[\frac{2}{5}, \frac{2}{3}\right]\right\}.$$

If the ball is know to be green acts f and g are equivalent. Otherwise the DM cannot compare (consider for instance $Q_1 = (\frac{2}{5}, \frac{3}{5}, 0)$ and $Q_2 = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$), i.e., f and g are not comparable w.r.t. $\succsim_{RB}^*$ (i.e., no unanimity).

Dynamic (in)Consistency (cont'd)

Example 1 (cont'd)

If we use the precautionary principle to complete $\succsim^*_{RB}$:

$$\min_{Q \in \mathcal{C}^{RB}} \mathbb{E}_Q[u(f)] = \frac{2}{5}u(10) > \frac{1}{3}u(10) = \min_{Q \in \mathcal{C}^{RB}} \mathbb{E}_Q[u(g)],$$

i.e. $f \succ^{\#}_{RB} g$ and Dynamic Consistency would imply $f = fRBg \succsim^{\#} g$, a contradiction (we saw before that $g \succ^{\#} f$).

Revising Objective Rationality

- In the perspective of this paper, Dynamic Consistency is viewed as a property of preferences fundamentally related to rationality. Therefore it is reasonable to require it for subjectively rational preferences.
- What we do: we propose a **modification of the unanimity rule** $\succsim^*$ in order to avoid violation of Dynamic Consistency when the MEU completion $\succsim^\#$ is considered.

Revising Objective Rationality (cont'd)

- From now on we consider two Bewley preferences, $\succsim^*$ and $\succsim^{**}$, represented respectively by $(u, \mathcal{C})$ and $(\hat{u}, \hat{\mathcal{C}})$.
- We fix a finite partition $\mathcal{P} = \{E_1, \cdots, E_n\} \subseteq \Sigma$, such that $P(E_i) > 0$ for all $i \in \{1, \cdots, n\}$ and all $P \in \mathcal{C}$.
- Preferences $\succsim_E^*$ and $\succsim_E^{**}$ denotes the dynamically consistent updates of $\succsim^*$ and $\succsim^{**}$, respectively.
  - Therefore $\succsim_E^*$ and $\succsim_E^{**}$ are Bewley preferences represented by $(u, \mathcal{C}^E)$ and $(\hat{u}, \hat{\mathcal{C}}^E)$.

Revising Objective Rationality (cont'd)

- This modification should satisfy two desiderata:
  (i) No new information is added, therefore $\succsim^{**}$ cannot be more complete than $\succsim^{*}$. This means that we cannot reverse the rankings given by the previous group of experts, and that no new comparisons can arise.
  (ii) The new preference $\succsim^{**}$ must coincide with the old one whenever violations of Dynamic Consistency do not occur.

Revising Objective Rationality (cont'd)

- The first desideratum brings us to the definition of a reassessment.

Definition 1

We say that $\succsim^{**}$ is a **reassessment** of $\succsim^{*}$ if for all $f, g \in \mathcal{F}$,

$$f \succsim^{**} g \Rightarrow f \succsim^{*} g$$

Revising Objective Rationality (cont'd)

- The second desideratum give us the following two axioms on the pair of preferences $(\succsim^*, \succsim^{**})$ that represents the main behavioral novelty of the paper.

(i) **Ex-ante coherence**: For all $x, x', y \in X$, for all $E \in \mathcal{P}$,

$$y \succsim^* xEx' \Rightarrow y \succsim^{**} xEx' \text{ and } xEx' \succsim^* y \Rightarrow xEx' \succsim^{**} y.$$

(ii) **Ex-post coherence**: For all $f, g \in \mathcal{F}$, for all $E \in \mathcal{P}$,

$$f \succsim_E^* g \Rightarrow f \succsim_E^{**} g.$$

Coherence Precautionary Reassessment

Definition 2

We say that $\succsim^{**}$ is the **coherent precautionary reassessment** of $\succsim^{*}$, if $\succsim^{**}$ is the most incomplete coherent reassessment of $\succsim^{*}$.

- Note that if we do not require $\succsim^{**}$ to be the most incomplete coherent reassessment, we can still get dynamic inconsistencies.

Additional results

Coherence Precautionary Reassessment (cont'd)

Main Theorem

The following assertions are equivalent:

(i) The preference $\succsim^{**}$ is the coherent precautionary reassessment of $\succsim^{*}$;

(ii) For all $f, g \in \mathcal{F}$,

$$f \succsim^{**} g \Leftrightarrow \sum_{i=1}^{n} P_0(E_i)\mathbb{E}_{P_i^{E_i}}[u(f)] \geq \sum_{i=1}^{n} P_0(E_i)\mathbb{E}_{P_i^{E_i}}[u(g)],$$

$$\forall P_0, P_1, \ldots, P_n \in \mathcal{C}.$$

Proposition 5 and Lemma 2

Coherence Precautionary Reassessment (cont'd)

- According to this decision criterion, this is *as if* we add "new" experts with veto power. The novel priors that are added are "obtained" from the old ones in the following way.

$$\underbrace{\sum_{i=1}^{n} P_0(E_i)}_{\text{cvx comb.}} \underbrace{\mathbb{E}_{P_i^{E_i}}[u(f)]}_{\substack{\text{EU of } f \text{ w.r.t.} \\ \text{update } P_i^{E_i}}}$$

1. Fix $n+1$ probabilities in $P_0, P_1, \cdots, P_n \in \mathcal{C}$.
2. Compute expected utility of f using the Bayesian update w.r.t. E_i of the corresponding probability P_i.
3. Expected utilities are aggregated through a convex combinations in which weights are given by $P_0(E_i)$.

Coherence Precautionary Reassessment (cont'd)

Definition 3

The rectangular hull of a set of priors $\mathcal{C} \subseteq \Delta$ w.r.t. partition $\mathcal{P}$ is given by given by

$$r_{\mathcal{P}}(\mathcal{C}) := \left\{ \sum_{i=1}^{n} P_0(E_i) \cdot P_i^{E_i} \middle| P_0, P_1, \cdots, P_n \in \mathcal{C} \right\}.$$

We say that a set $\mathcal{C} \subseteq \Delta$ is rectangular (w.r.t. $\mathcal{P}$) when $\mathcal{C} = r_{\mathcal{P}}(\mathcal{C})$.

- Defined previously by Sarin and Wakker (1998) and Epstein and Schneider (2003).

Proposition 5 and Lemma 2

Coherence Precautionary Reassessment (cont'd)

Corollary 1

Item (ii) of Theorem 2 is equivalent to

(iii) For all $f, g \in \mathcal{F}$,

$$f \succsim^{**} g \Leftrightarrow \mathbb{E}_Q[u(f)] \geq \mathbb{E}_Q[u(g)], \ \forall Q \in r_{\mathcal{P}}(\mathcal{C}).$$

Dynamic Consi stent Subjective Preferences

Corollary 2

Let $\succsim^{**}$ be the coherent precautionary reassessment of $\succsim^{*}$. If we take the MEU precautionary completion of $\succsim^{**}$ then the pair $(\succsim^{\#\#}, \succsim_E^{\#\#})$ satisfies Dynamic Consistency.

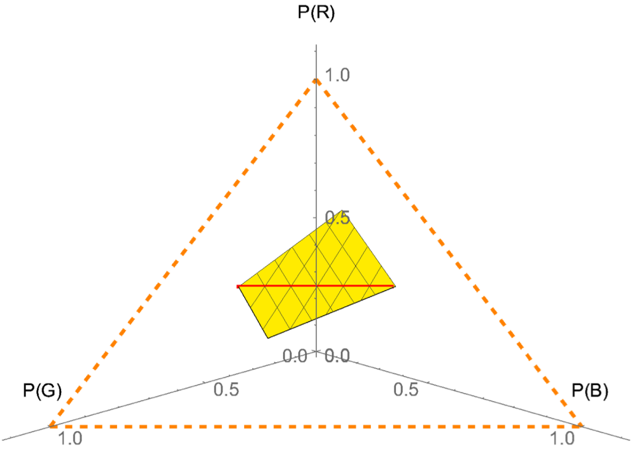
Dynamic Consistent Subjective Preferences (cont'd)

Example 1 - (cont'd)

Now we compute the coherent precautionary reassessment $\succsim^{**}$ w.r.t. $\succsim^{*}$ given the partition $\mathcal{P}$. Its a Bewley preference represented by $(u, r_{\mathcal{P}}(\mathcal{C}))$ with

$$r_{\mathcal{P}}(\mathcal{C}) = \left\{ (1-p) \begin{pmatrix} q \\ 1-q \\ 0 \end{pmatrix} + p \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| p \in \left[\frac{1}{6}, \frac{1}{2}\right], q \in \left[\frac{2}{5}, \frac{2}{3}\right] \right\}.$$

Dynamic Consistent Subjective Preferences (cont'd)



Dynamic Consistent Subjective Preferences (cont'd)

Example 1 - (cont'd)

Acts f and g are not comparable with respect to $\succsim^{**}$.

Computing $\succsim^{\#\#}$ obtained by Corollary 2:

$$I(g) = \min_{p \in [\frac{1}{6}, \frac{1}{2}],\, q \in [\frac{2}{5}, \frac{2}{3}]} u(10)[(1-p)(1-q) - p] = \frac{4}{9} u(10)$$

$$I(f) = \min_{p \in [\frac{1}{6}, \frac{1}{2}],\, q \in [\frac{2}{5}, \frac{2}{3}]} u(10)[(1-p)q + p] = \frac{1}{2} u(10)$$

Hence $f \succ^{\#\#} g$ and no Dynamic Consistency problem arise. Actually we obtain: $f \succsim^{\#\#} g$, $f \succsim^{\#\#}_{RB} g$ and $f \succsim^{\#\#}_{G} g$.

Conclusion

- The **unanimity** rule says that a policy f is preferred to g if, and only if, every expert (or equivalently probability measure) assigns higher expected utility to f rather than g.
  - $*$ If two experts disagree, this rule is unable to tell which policy is better.
- When a decision must be taken, several authors suggest to compare policies through the **precautionary principle**: the policy with the highest minimal expected utility should be chosen.

Conclusion

- This rule may generate possible **dynamic inconsistencies** when an intermediary period of partial resolution of uncertainty is added.
- In order to avoid that, we provide axioms that modify the original group of experts.
  - We derive a new unanimity rule called the **coherent precautionary reassessment**.
  - New opinions are formed by taking convex combinations of experts' updated beliefs.
  - This makes the completion of the new unanimity rule dynamically consistent.

Thank you!

References

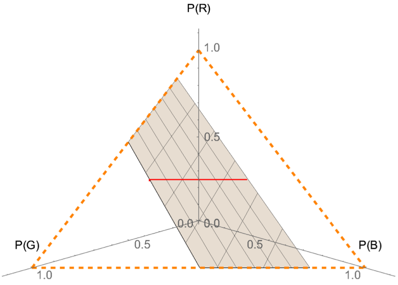
1. Bewley, T., 2002. Knightian decision theory: Part I. *Decisions in Economics and Finance*, 25(2), 79-110.
2. Ellsberg, D., 1961. Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics* 75(4), 643-669.
3. Epstein, L.G. and Schneider, M., 2003. Recursive multiple-priors. *Journal of Economic Theory*, 113(1), 1-31.
4. Gilboa, I., 2009. Theory of Decision under Uncertainty, Cambridge University Press, 2009.
5. Gilboa, I., Maccheroni, F., Marinacci, M., and Schmeidler, D., 2010. Objective and subjective rationality in a multiple prior model. *Econometrica*, 78(2), 755-770.
6. Gilboa, I., and Schmeidler, D., 1989. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18, 141-153.
7. Sarin, R., and Wakker, P., 1998. Dynamic choice and nonexpected utility. *Journal of Risk and Uncertainty* 17, 87-119.

Additional Results

Proposition 2

$\succsim^1$ is the **ex-ante-coherent precautionary reassessment** of $\succsim^*$ if and only if $\succsim^1$ is represented by $(u, \mathcal{C}_1)$.

$$\mathcal{C}_1 = \{Q \in \Delta : \exists P \in \mathcal{C} \text{ s.t. } P(E) = Q(E), \forall E \in \mathcal{P}\}$$

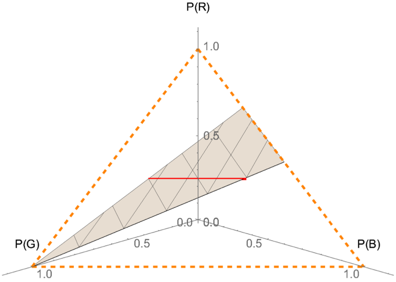


Additional Results (cont'd)

Proposition 3

$\succsim^2$ is the **ex-post-coherent precautionary reassessment** of $\succsim^*$ if and only if $\succsim^2$ is represented by $(u, \mathcal{C}_2)$.

$$\mathcal{C}_2 = \{Q \in \Delta | \forall i = 1, \ldots, n \, \exists P_i \in \mathcal{C} \text{ s.t. } P_i^{E_i} = Q^{E_i}\}$$



Back presentation

Proposition 5 and Lemma 2

Proposition 5

$\succsim^{**}$ is a coherent reassessment of $\succsim^{*}$ if and only if

(i) The affine utility functions $\hat{u}$ and u represent the same preference over X;

(ii) $\hat{\mathcal{C}} \supseteq \mathcal{C}$;

(iii) For all $E \in \mathcal{P}$, $\mathcal{C}^E = \hat{\mathcal{C}}^E$;

(iv) For all $Q \in \hat{\mathcal{C}}$ there exists $P \in \mathcal{C}$ s.t. $P(E) = Q(E)$, for all $E \in \mathcal{P}$.

Lemma 2

$r_{\mathcal{P}}(\mathcal{C})$ is the maximal set such that

(i) $r_{\mathcal{P}}(\mathcal{C})^E = \mathcal{C}^E$ for all $E \in \mathcal{P}$;

(ii) $\forall Q \in r_{\mathcal{P}}(\mathcal{C})$, $\exists P \in \mathcal{C}$ such that $P(E) = Q(E)$, $\forall E \in \mathcal{P}$.

Back presentation

Completion of a Bewley preference by a MEU

Theorem 1 (GMMS, Theorem 4)

Let $\succsim^*$ be a Bewley preference represented by $(u, \mathcal{C})$ and let $\succsim^\#$ be a complete and continuous preorder. Then:

(i) The pair $(\succsim^*, \succsim^\#)$ jointly satisfies Consistency and Default to Certainty;

(ii) $\succsim^\#$ is a Maxmin preference represented by $(u, \mathcal{C})$, that is, for all $f, g \in \mathcal{F}$:

$$f \succsim^\# g \Leftrightarrow \min_{P \in \mathcal{C}} \mathbb{E}_P[u(f)] \geq \min_{P \in \mathcal{C}} \mathbb{E}_P[u(g)].$$

Axioms GMMS Back presentation

GMMS: Axioms

- Reflexive: For all $f \in \mathcal{F}$, $f \succsim f$.
- Transitive: For all $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$, then $f \succsim h$.
- Mixture continuity: For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$ are closed in $[0,1]$. are closed in $[0,1]$.
- C-Completeness: For all $x, y \in X$, $x \succsim^* y$ or $y \succsim^* x$.
- Completeness: For all $f, g \in \mathcal{F}$, $f \succsim^\# g$ or $g \succsim^\# f$.
- Independence: For all $f, g, h \in \mathcal{F}$, $\forall \alpha \in [0,1]$,

$$f \succsim^* g \text{ iff } \alpha f + (1-\alpha) h \succsim^* \alpha g + (1-\alpha) h.$$

- Consistency: For all $f, g \in \mathcal{F}$, $f \succsim^* g \Rightarrow f \succsim^\# g$.
- Default to Certainty: For all $f \in \mathcal{F}$ and $x \in X$,

$$\text{if not } f \succsim^* x \Rightarrow x \succ^\# f.$$

Back presentation