Integral quantizations: Weyl-Heisenberg versus affine group

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[1] H Bergeron and J.-P G *Integral quantizations with two basic examples* (2013), arXiv:1308.2348 [quant-ph]

[2] S.T. Ali, J.-P Antoine, and J.-P G *Coherent States, Wavelets and their Generalizations* (Graduate Texts in Mathematics, Springer, New York, 2000), 2nd Edition to be published, 2013.

[3] H. Bergeron, E. M. F. Curado, J.-P.G. and Ligia M. C. S. Rodrigues, *Integral quantization: Weyl-Heisenberg versus affine group*, to be published in Proceedings of the 8th Symposium on Quantum Theory and Symmetries, El Colegio Nacional, Mexico City, 5-9 August, 2013, Ed. K.B. Wolf, J. Phys.: Conf. Ser. (2013)

[4] H Bergeron, A Dapor, J-P G and P Małkiewicz, *Wavelet Quantum Cosmology* (2013); arXiv:1305.0653 [gr-qc]

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Theoretical and Mathematical Physics

S. T. Ali · J-P Antoine · J-P. Gazeau

Coherent States, Wavelets, and Their Generalizations

Second Edition

This second edition is fully updated, covering in particular new types of coherent states (the so-called Gazeau-Klauder coherent states, nonlinear coherent states, squeezed states, as used now routinely in quantum optics) and various generalizations of wavelets (wavelets on manifolds, curvelets, shearlets, etc.). In addition, it contains a new chapter on coherent state quantization and the related probabilistic aspects. As a survey of the theory of coherent states, wavelets, and some of their generalizations, it emphasizes mathematical principles, subsuming the theories of both wavelets and coherent states into a single analytic structure. The approach allows the user to take a classical-like view of quantum states in physics.

Starting from the standard theory of coherent states over Lie groups, the authors generalize the formalism by associating coherent states to group representations that are square integrable over a homogeneous space; a further step allows one to dispense with the group context altogether. In this context, wavelets can be generated from coherent states of the affine group of the real line, and higher-dimensional wavelets arise from coherent states of other groups. The unified background makes transparent an entire range of properties of wavelets and coherent states. Many concrete examples, such as coherent states from semisimple Lie groups, Gazeau-Klauder coherent states, coherent states for the relativity groups, and several kinds of wavelets, are discussed in detail. The book concludes with a palette of potential applications, from the quantum physically oriented, like the quantum-classical transition or the construction of adequate states in quantum information, to the most innovative techniques to be used in data processing.

Intended as an introduction to current research for graduate students and others entering the field, the mathematical discussion is self-contained. With its extensive references to the research literature, the first edition of the book is already a proven compendium for physicists and mathematicians active in the field, and with full coverage of the latest theory and results the revised second edition is even more valuable.

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Theoretical and Mathematical Physics

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Coherent States, Wavelets, and Their Generalizations

Second Edition

Springer

1. What is really quantization?

What is really ...?

- In digital signal processing: *quantization* maps a large set of input values to a smaller set such as rounding values to some unit of precision. Typically, a change of scale.

- In physics or mathematics, the term has a different meaning. For instance, the perplexing "*Quantization can be any procedure that associates a quantum mechanical observable to a given classical dynamical variable*".[a]

- Or even more perplexing: " *First quantization is a mystery. It is the attempt to get from a classical description of a physical system to a quantum description of the "same" system. Now it doesn't seem to be true that God created a classical universe on the first day and then quantized it on the second day...*"[b]

- Or the following: " *We quantize things we do not really know to obtain things most of which we are unable to measure*"[c]

- The basic procedure, named "canonical", starting from a phase space or symplectic manifold

$$\mathbb{R}^2 \ni (q,p)\,, \quad \{q,p\} = 1 \mapsto (Q,P)\,, \quad [Q,P] = i\hbar I\,,$$
$$f(q,p) \mapsto f(Q,P) \mapsto (\mathrm{Sym} f)(Q,P)\,.$$

- But then what about singular f, e.g. the phase $\arctan(p/q)$? What about barriers or other impassable boundaries? The motion on a circle? In a bounded interval? On the half-line? ....

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[a] J. Kiukas, P. Lahti, and K. Ylinenc, Phase space quantization and the operator moment problem, *J. Math. Phys.* **47** 072104 (2006).

[b] J. Baez, Categories, quantization and much more, http://math.ucr.edu/home/baez/categories.html (2006)

[c] J.P.G., Metrobus Gavea-Botafogo 04/09/2013 morning


What about integral quantization??
What about POVM??
Quantization MUST be CANONICAL !!

More mathematically precise:

- Quantization is

  (i) a linear map

$$\mathfrak{Q} : \mathcal{C}(X) \mapsto \mathcal{A}(\mathfrak{H})$$

  $\mathcal{C}(X)$: vector space of complex-valued functions $f(x)$ on a set X
  $\mathcal{A}(\mathfrak{H})$: vector space of linear operators

$$\mathfrak{Q}(f) \equiv A_f$$

  in some complex Hilbert space $\mathfrak{H}$ such that

  (ii) to $f = 1 \longrightarrow$ identity operator I on $\mathfrak{H}$,

  (iii) to real f there $\longrightarrow$ (essentially) self-adjoint operator A_f in $\mathfrak{H}$.

- Add further requirements on X and $\mathcal{C}(X)$ (e.g., measure, topology, manifold, closure under algebraic operations...)

- Add physical interpretation about measurement of spectra of classical $f \in \mathcal{C}(X)$ or quantum $\mathcal{A}(\mathfrak{H})$ to which are given the status of *observables*.

- Add requirement of unambiguous classical limit of the quantum physical quantities, the limit operation being associated to a change of scale

2. Integral quantization

Integral quantization: general setting and POVM

- (X, ν): measure space.
- $X \ni x \mapsto \mathsf{M}(x) \in \mathcal{L}(\mathfrak{H})$: X-labelled family of bounded operators on Hilbert space $\mathfrak{H}$ resolving the identity I:

$$\int_X \mathsf{M}(x)\, \mathrm{d}\nu(x) = I\,, \quad \text{in a weak sense} \tag{1}$$

- If the $\mathsf{M}(x)$'x are positive and unit trace,

$$\mathsf{M}(x) \equiv \rho(x) \quad \text{(density matrix)}$$

- If X is space with suitable topology, the map

$$\mathcal{B}(X) \ni \Delta \mapsto \int_\Delta \rho(x)\, \mathrm{d}\nu(x)$$

may define a normalized positive operator-valued measure (POVM) on the σ-algebra $\mathcal{B}(X)$ of Borel sets.

Integral quantization: the map

- Quantization of complex-valued functions $f(x)$ on X is the linear map:

$$f \mapsto A_f = \int_X \mathsf{M}(x)\, f(x)\, \mathrm{d}\nu(x)\,, \tag{2}$$

- understood as the sesquilinear form,

$$B_f(\psi_1, \psi_2) = \int_X \langle \psi_1 | \mathsf{M}(x) | \psi_2 \rangle\, f(x)\, \mathrm{d}\nu(x)\,, \tag{3}$$

defined on a dense subspace of $\mathfrak{H}$.

- If f is real and at least semi-bounded, the Friedrich's extension of B_f univocally defines a self-adjoint operator.
- If f is not semi-bounded, no natural choice of a self-adjoint operator associated with B_f, a subtle question[a]. We need more information on $\mathcal{H}$.

[a] see for instance H. Bergeron, JPG, P. Siegl, A. Youssef, Eur. Phys. Lett. **92** 60003 (2010); H. Bergeron, P. Siegl, A. Youssef, J. Phys. A: Math. Theor. **45** 244028 (2012)

Integral quantization: back to classical

- If $\mathsf{M}(x) = \rho(x)$ and with another (or the same) family of positive unit trace operators $X \ni x \mapsto \widetilde{\rho}(x) \in \mathcal{L}^+(\mathfrak{H})$ go back to the classical

$$A_f \mapsto \check{f}(x) := \int_X \mathrm{tr}(\widetilde{\rho}(x)\rho(x'))\, f(x')\, \mathrm{d}\nu(x')\,, \text{ “lower symbol”} \tag{4}$$

  provided the integral be defined.

- Then classical limit condition means: given a scale parameter ϵ and a distance $d(f, \check{f})$:

$$d(f, \check{f}) \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{5}$$

Integral quantization: comments

- Quantization issues, e.g. spectral properties of A_f, may be derived from functional properties of the *lower symbol* $\check{f}$.

- Quantizing constraints: suppose that (X, ν) is a smooth n-dim. manifold on which is defined space $\mathcal{D}'(X)$ of distributions as the topological dual of compactly supported n-forms on X. Some of these distributions, e.g. $\delta(u(x))$, express geometrical constraints. Extending the map $f \mapsto A_f$ to these objects yields the quantum version $A_{\delta(u(x))}$ of these constraints.

- Different starting point, more in Dirac's spirit[a] (e.g. see (Loop) Quantum Gravity and Quantum Cosmology) would consist in determining the kernel of the operator A_u issued from integral quantization $u \mapsto A_u$.

- Both methods are obviously not *mathematically* equivalent, except for a few cases. They are possibly *physically* equivalent.

[a] P.A.M. Dirac, Lectures on Quantum Mechanics, Dover, New York, 2001

Covariant integral quantization with UIR of a group

- Let G be a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation (UIR) of G in a Hilbert space $\mathfrak{H}$.

- Let M be a bounded operator on $\mathfrak{H}$. Suppose that the operator

$$R := \int_G \mathsf{M}(g)\, \mathrm{d}\mu(g)\,, \quad \mathsf{M}(g) := U(g)\, \mathsf{M}\, U^{\dagger}(g)\,, \tag{6}$$

  is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$ we have $U(g_0)\, R\, U^{\dagger}(g_0) = \int_G \mathrm{d}\mu(g)\, \mathsf{M}(g_0 g) = R$ and so R commutes with all operators $U(g)$, $g \in G$. Thus, from Schur's Lemma, $R = c_{\mathsf{M}} I$ with

$$c_{\mathsf{M}} = \int_G \mathrm{tr}\left(\rho_0\, \mathsf{M}(g)\right)\, \mathrm{d}\mu(g)\,, \tag{7}$$

  where the unit trace positive operator ρ_0 is chosen in order to make the integral convergent.

- Resolution of the identity follows:

$$\int_G \mathsf{M}(g)\, \mathrm{d}\nu(g) = I\,, \quad \mathrm{d}\nu(g) := \mathrm{d}\mu(g)/c_{\mathsf{M}}\,. \tag{8}$$

Covariant integral quantization: with square integrable UIR (e.g. affine group)

- For square-integrable UIR U for which $|\eta\rangle$ is an admissible unit vector, i.e. $c(\eta) := \int_G \mathrm{d}\mu(g)\, |\langle\eta|U(g)\eta\rangle|^2 < \infty$.
- Resolution of the identity is obeyed by *coherent states* for G:

$$|\eta_g\rangle = U(g)|\eta\rangle \quad \text{or by} \quad |\eta_g\rangle\langle\eta_g| = \rho(g)\,, \quad \rho := |\eta\rangle\langle\eta|$$

- This allows *covariant* integral quantization of complex-valued functions on the group $f \mapsto A_f = \int_G \rho(g)\, f(g)\, \mathrm{d}\nu(g)$:

$$U(g) A_f U^{\dagger}(g) = A_{U_r(g) f}\,, \tag{9}$$

  With $f \in L^2(G, \mathrm{d}\mu(g))$, $(U_r(g)f)(g') := f(g^{-1}g')$ is the regular representation.

- Generalization of the Berezin or heat kernel transform on G: $\check{f}(g) := \int_G \mathrm{tr}(\rho(g)\,\rho(g'))\, f(g')\, \mathrm{d}\nu(g')$.

Covariant quantization with UIR square integrable w.r.t. a subgroup (e.g. Weyl Heisenberg group)

- In the absence of square-integrability over G, there exists a definition of square-integrable covariant coherent states with respect to a left coset manifold $X = G/H$, with H a closed subgroup of G, equipped with a quasi-invariant measure ν.[a]

[a] S. T. Ali, J.-P. Antoine, and J.-P. G., *Coherent States, Wavelets and their Generalizations* (Graduate Texts in Mathematics, Springer, New York, 2000). New edition in 2014

3. Weyl-Heisenberg covariant integral quantization(s)

Weyl-Heisenberg algebra and its Fock or number representation

- Let $\mathcal{H}$ be a separable (complex) Hilbert space with orthonormal basis $e_0, e_1, \dots, e_n \equiv |e_n\rangle, \dots,$ (e.g the Fock space with $|e_n\rangle \equiv |n\rangle$).

- Lowering and raising operators a and $a^\dagger$:

$$a\,|e_n\rangle = \sqrt{n}|e_{n-1}\rangle\,, \quad a|e_0\rangle = 0\,,$$
$$a^\dagger\,|e_n\rangle = \sqrt{n+1}|e_{n+1}\rangle\,.$$

- Operator algebra $\{a, a^\dagger, 1\}$ is defined by

$$[a, a^\dagger] = 1\,.$$

- Number operator: $N = a^\dagger a$, spectrum $\mathbb{N}$, $N|e_n\rangle = n|e_n\rangle$.

Unitary Weyl-Heisenberg group representation and standard CS

- Square integrability holds with respect to center C of the Weyl-Heisenberg group and $X = G_{\mathrm{WH}}/C \sim \mathbb{C}$ with measure $\mathrm{d}^2 z/\pi$.
- To each $z \in \mathbb{C}$ corresponds the (unitary) displacement operator $D(z)$:

$$\mathbb{C} \ni z \mapsto D(z) = e^{za^\dagger - \bar{z}a} \, .$$

- Space inversion $\rightarrow$ Unitarity:

$$D(-z) = (D(z))^{-1} = D(z)^\dagger \, .$$

- Addition formula (Quantum Mechanics in a nutshell!):

$$D(z)D(z') = e^{\frac{1}{2}(z\bar{z}' - \bar{z}z')} D(z + z') = e^{(z\bar{z}' - \bar{z}z')} D(z')D(z) \, ,$$

- Standard (i.e., Schrödinger-Klauder-Glauber-Sudarshan) CS

$$|z\rangle = D(z)|e_0\rangle \, ,$$

Quantization(s) with weight function(s) I

- Let $\varpi(z)$ be a function on the complex plane obeying $\varpi(0) = 1$. Suppose that it allows to define a bounded operator M on $\mathfrak{H}$ through the operator-valued integral
$$\mathsf{M} = \int_{\mathbb{C}} \varpi(z) D(z) \, \frac{\mathrm{d}^2 z}{\pi} \, .$$
- Then, the family of displaced $\mathsf{M}(z) := D(z)\mathsf{M}D(z)^\dagger$ under the unitary action $D(z)$ resolves the identity
$$\int_{\mathbb{C}} \mathsf{M}(z) \, \frac{\mathrm{d}^2 z}{\pi} = I \, .$$
- It is a direct consequence of $D(z)D(z')D(z)^\dagger = e^{z\overline{z}' - \overline{z}z'} D(z')$, of $\int_{\mathbb{C}} e^{z\bar{\xi} - \bar{z}\xi} \, \frac{\mathrm{d}^2 \xi}{\pi} = \pi \delta^2(z)$, and of $\varpi(0) = 1$ with $D(0) = I$.

Quantization(s) with weight function(s) II

- The resulting quantization map is given by

$$f \mapsto A_f = \int_{\mathbb{C}} \mathsf{M}(z)\, f(z)\, \frac{\mathrm{d}^2 z}{\pi}\,.$$

- Equivalently $A_f = \int_{\mathbb{C}} \varpi(z)\, D(z)\, \hat{f}(-z)\, \frac{\mathrm{d}^2 z}{\pi}$, where is involved the symplectic Fourier transform $\hat{f}(z) = \int_{\mathbb{C}} e^{z\bar{\xi} - \bar{z}\xi} f(\xi)\, \frac{\mathrm{d}^2 \xi}{\pi}$
- Covariance:

$$A_{f(z-z_0)} = D(z_0) A_{f(z)} D(z_0)^{\dagger}\,.$$

- Properties:

$$A_{f(-z)} = \mathsf{P} A_{f(z)} \mathsf{P}, \forall\, f \quad \Longleftrightarrow \quad \varpi(z) = \varpi(-z), \,\forall\, z\,,$$
$$A_{\overline{f(z)}} = A^{\dagger}_{f(z)}, \forall\, f \quad \Longleftrightarrow \quad \overline{\varpi(-z)} = \varpi(z), \,\forall\, z\,,$$

where $\mathsf{P} = \sum_{n=0}^{\infty} (-1)^n |e_n\rangle\langle e_n|$ is the parity operator.

CCR is (almost always) the rule!

- The quantization map $f \mapsto A_f$ yields the canonical commutation rule

$$[a, a^\dagger] = I$$

for all even real weight function ϖ.

- Indeed

$$A_z = a\,, \quad A_{\overline{f(z)}} = A^\dagger_{f(z)}\,.$$

- Equivalently, with $z = (q + ip)/\sqrt{2}$,

$$A_q = \frac{a + a^\dagger}{\sqrt{2}} := Q\,,\ A_p = \frac{a - a^\dagger}{i\sqrt{2}} := P\,,\ [Q, P] = iI$$

- Moreover, if $|\varpi(z)| = 1$

$$\mathrm{tr}(A^\dagger_f A_f) = \int_{\mathbb{C}} |f(z)|^2 \frac{\mathrm{d}^2 z}{\pi}\,,$$

which means that the map $f \mapsto A_f$ is invertible through a trace formula.

Wigner-Weyl, CS, normal, and other, quantizations

- The normal, Wigner-Weyl and anti-normal (i.e., anti-Wick or Berezin or CS) quantizations correspond to $s \to 1_-$, $s = 0$, $s = -1$ resp. in the specific choice [a]

$$\varpi_s(z) = e^{s|z|^2/2}, \quad \mathrm{Re}\, s < 1.$$

- This yields a diagonal $\mathsf{M} \equiv \mathsf{M}_s$ with

$$\langle e_n|\mathsf{M}_s|e_n\rangle = \frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^n,$$

and so

$$\mathsf{M}_s = \int_{\mathbb{C}} \varpi_s(z) D(z)\, \frac{\mathrm{d}^2 z}{\pi} = \frac{2}{1-s} \exp\left[\ln\left(\frac{s+1}{s-1}\right) a^\dagger a\right].$$

[a] K.E. Cahill and R. Glauber, Ordered expansion in Boson Amplitude Operators, *Phys. Rev.* **117** 1857-1881 (1969)

Wigner-Weyl, CS, normal, and other, quantizations II

- The case $s = -1$ corresponds to the CS (anti-normal) quantization, since

$$\mathsf{M} = \lim_{s \to -1} \frac{2}{1-s} \exp\left(\ln \frac{s+1}{s-1} a^\dagger a \right) = |e_0\rangle\langle e_0| \,,$$

  and so

$$A_f = \int_{\mathbb{C}} D(z) \mathsf{M} D(z)^\dagger \, f(z) \, \frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} |z\rangle\langle z| \, f(z) \, \frac{\mathrm{d}^2 z}{\pi} \,.$$

- The choice $s = 0$ implies $\mathsf{M} = 2\mathsf{P}$ and corresponds to the Wigner-Weyl quantization. Then

$$A_f = \int_{\mathbb{C}} D(z) \, 2\mathsf{P} \, D(z)^\dagger \, f(z) \, \frac{\mathrm{d}^2 z}{\pi} \,.$$

- The case $s = 1$ is the normal quantization in an asymptotic sense.

- The parameter s was originally introduced by Cahill and Glauber in view of discussing the problem of expanding an arbitrary operator as an ordered power series in a and $a^\dagger$, a typical question encountered in quantum field theory, specially in quantum optics. Actually, they were not interested in the question of quantization itself.

Canonical quantization with POVM or not

- Operator M_s is positive unit trace class for $s \leqslant -1$ (and only trace class if $\mathrm{Re}\ s < 0$), i.e., is density operator: quantization has a consistent probabilistic content, the operator-valued measure

$$\mathbb{C} \supset \Delta \mapsto \int_{\Delta \in \mathcal{B}(\mathbb{C})} D(z) \mathsf{M}_s D(z)^\dagger \frac{\mathrm{d}^2 z}{\pi},$$

is a positive operator-valued measure.

- Given an elementary quantum energy, say $\hbar\omega$ and with the temperature T-dependent $s = -\coth \dfrac{\hbar\omega}{2k_B T}$ the density operator quantization is Boltzmann-Planck

$$\rho_s = \left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right) \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{k_B T}} |e_n\rangle\langle e_n|\,.$$

- Interestingly, the temperature-dependent operators $\rho_s(z) = D(z)\,\rho_s\,D(z)^\dagger$ defines a Weyl-Heisenberg covariant family of POVM's on the phase space $\mathbb{C}$, the null temperature limit case being the POVM built from standard CS.

Quantum harmonic oscillator according to ϖ

- For real even ϖ, $A_{q^2} = Q^2 - \partial_z\partial_{\bar z}\varpi|_{z=0} + \frac{1}{2}\left(\partial_z^2\varpi|_{z=0} + \partial_{\bar z}^2\varpi|_{z=0}\right)$, $A_{p^2} = P^2 - \partial_z\partial_{\bar z}\varpi|_{z=0} - \frac{1}{2}\left(\partial_z^2\varpi|_{z=0} + \partial_{\bar z}^2\varpi|_{z=0}\right)$ and so

$$A_{|z|^2} \equiv A_J = a^\dagger a + \frac{1}{2} - \partial_z\partial_{\bar z}\varpi|_{z=0}\ .$$

  where $|z|^2 (= J)$ is the energy (or action variable) for the H.O.

- The difference between the ground state energy $E_0 = 1/2 - \partial_z\partial_{\bar z}\varpi|_{z=0}$, and the minimum of the quantum potential energy $E_m = [\min(A_{q^2}) + \min(A_{p^2})]/2 = -\partial_z\partial_{\bar z}\varpi|_{z=0}$ is independent of the particular (regular) quantization chosen, namely $E_0 - E_m = 1/2$ (experimentally verified in 1925).

- In the exponential Cahill-Glauber case $\varpi_s(z) = e^{s|z|^2/2}$ the above operators reduce to

$$A_{|z|^2} = a^\dagger a + \frac{1-s}{2}\,, A_{q^2} = Q^2 - \frac{s}{2} A_{p^2} = P^2 - \frac{s}{2}\,.$$

- It has been proven [a] that these constant shifts in energy are inaccessible to measurement.

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[a] H. Bergeron, J.P. G., A. Youssef, Are the Weyl and coherent state descriptions physically equivalent?, Physics Letters A 377 (2013) 598605

Weyl-Heisenberg integral quantization with action-angle variables

- With $z = \sqrt{J}\, e^{i\gamma}$ in action-angle (J, γ) notations for the harmonic oscillator, quantization of $f(J, \gamma)$, 2π-periodic in γ, yields formally

$$A_f = \int_0^{+\infty} \mathrm{d}J \int_0^{2\pi} \frac{\mathrm{d}\gamma}{2\pi} f(J, \gamma) \rho \left(\sqrt{J} e^{i\gamma}\right) . \tag{10}$$

- Define the unitary representation $\theta \mapsto U_{\mathbb{T}}(\theta)$ of the unit circle $\mathbb{S}^1$ on the Hilbert space $\mathcal{H}$ as $U_{\mathbb{T}}(\theta)|e_n\rangle = e^{i(n+\nu)\theta}|e_n\rangle$, where ν is arbitrary real. One verifies, in the case of diagonal ρ, the angular covariance property:

$$U_{\mathbb{T}}(\theta) A_f U_{\mathbb{T}}(-\theta) = A_{T(\theta)f}\,, \quad T(\theta) f(J, \gamma) = f(J, \gamma - \theta)\,. \tag{11}$$

CS quantization of discontinuous functions: the quantum angle[a] or phase

- As an example, let us quantize with coherent states, $\rho(z) = |z\rangle\langle z|$, the discontinuous 2π-periodic angle function $\daleth(\gamma) = \gamma$ for $\gamma \in [0, 2\pi)$.
- In terms of the action-angle variables standard CS read as

$$|z\rangle \equiv |J, \gamma\rangle = \sum_n \sqrt{p_n(J)} e^{in\gamma} |e_n\rangle \,, \tag{12}$$

where $n \mapsto p_n(J) = e^{-J} J^n / n!$ is the Poisson distribution.

- The action variable is precisely the Poisson average of the discrete variable n, $\langle n\rangle_{\text{poisson}} = J$. Note that in electromagnetism, the variables J and γ represent the field intensity and the phase, respectively.
- Since the angle function is real and bounded, its quantum counterpart $A_{\daleth}$ is a bounded self-adjoint operator, and it is covariant in the above sense.

[a] JPG, F. Szafraniec, Quantum angle operator, in progress

Quantum phase and its classical portrait

- In the basis $|e_n\rangle$, quantum phase or angle operator $A_{\daleth}$ is given by the infinite matrix:

$$A_{\daleth} = \pi\, 1_{\mathcal{H}} + i \sum_{n\neq n'} \frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\sqrt{n!n'!}} \frac{1}{n'-n} |e_n\rangle\langle e_{n'}| \,. \tag{13}$$

  This operator has spectral measure with support $[0, 2\pi]$.

- The corresponding "lower symbol" reads as the Fourier sine series:

$$\langle J,\gamma|A_{\daleth}|J,\gamma\rangle = \pi - 2\sum_{q=1}^{\infty} d_q(\sqrt{J})\,\frac{\sin q\gamma}{q}\,,$$

  with $d_q(r) = e^{-r^2} r^q \frac{\Gamma(\frac{q}{2}+1)}{\Gamma(q+1)}\, {}_1F_1(\frac{q}{2}+1; q+1; r^2)$ balances the trigonometric Fourier coefficient $2/q$ of the angle function $\daleth$. It can be shown [a] that this positive function is bounded by 1.

[a] JPG and M. del Olmo, *q-coherent states quantization of the harmonic oscillator,* Annals of Physics (NY) **330** 220-245 (2013); arXiv:1207.1200 [quant-ph]

Semi-classical behavior

- At small J, the lower symbol oscillates around its average value π with amplitude equal to $\sqrt{\pi J}$:

$$\langle J, \gamma | A_{\daleth} | J, \gamma \rangle \approx \pi - \sqrt{\pi J} \, \sin \gamma \, .$$

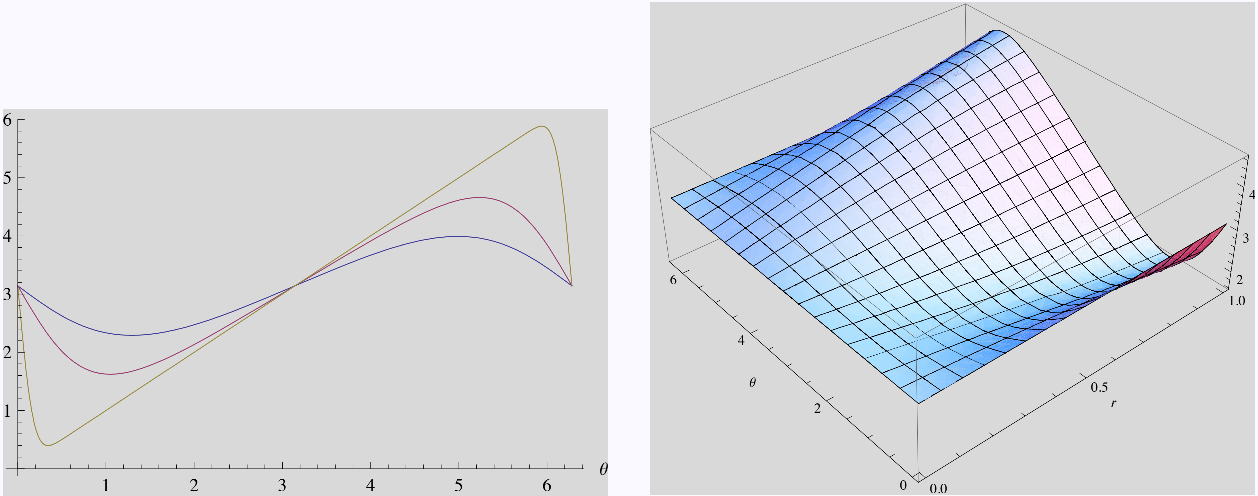
- At large J, we recover the Fourier series of the 2π-periodic angle function:

$$\langle J, \gamma | A_{\daleth} | J, \gamma \rangle \approx \pi - 2 \sum_{q=1}^{\infty} \frac{1}{q} \sin q\gamma = \daleth(\gamma) \quad \text{for} \quad \gamma \in [0, 2\pi) \, .$$

- By re-injecting physical dimensions, $|z|^2 = J$ is an action and should appear in the formulas as divided by $\hbar$: the limit $J \to \infty$ is the classical limit $\hbar \to 0$.

Lower symbol of the phase operator

Behavior of $\langle J, \gamma | A_{\gimel} | J, \gamma \rangle$ as a function of $\theta \equiv \gamma$ for different values of J. Observe how much it becomes close to the classical one at the largest value of J.



Lower symbol of the angle operator for $\sqrt{J} = \{0.5, 1, 5\}$ and $\gamma \equiv \theta \in [0, 2\pi)$ (left) and for $\left(\sqrt{J}, \gamma\right) \in [0, 1] \times [0, 2\pi)$ (right).

Semi-classical behavior continued

- The number operator $\hat{N} = a^\dagger a$ is, up to a constant shift, the quantization of the classical action, $A_J = \hat{N} + 1$: $A_J = \sum_n (n+1)|e_n\rangle\langle e_n|$.

- Are the commutator of action and angle operators and its lower symbol close to the canonical value iI?

$$[A_\daleth, A_J] = i \sum_{n \neq n'} \frac{\Gamma\left(\frac{n+n'}{2} + 1\right)}{\sqrt{n!n'!}} |n\rangle\langle n'| \,,$$

$$\langle J, \gamma | [A_\daleth, A_J] | J, \gamma \rangle = 2i \sum_{q=1}^{\infty} d_q(\sqrt{J}) \cos q\gamma \equiv i\,\mathcal{C}(J, \gamma) \,.$$

- At small J, the function $\mathcal{C}(J,\gamma)$ oscillates around 0 with amplitude equal to $\sqrt{\pi}\sqrt{J}$: $\mathcal{C}(J,\gamma) \approx \sqrt{\pi}\sqrt{J}\ \cos\gamma$. Applying the Poisson summation formula, we get at $J \to \infty$ (or $\hbar \to 0$) the expected "canonical" behavior for $\gamma \in [0, 2\pi)$:

$$\langle J, \gamma | [A_\daleth, A_J] | J, \gamma \rangle \approx -i + 2\pi i \sum_{n \in \mathbb{Z}} \delta(\gamma - 2\pi n) \,.$$

- At $J \to \infty$ the commutator symbol becomes canonical for $\gamma \neq 2\pi n$, $n \in \mathbb{Z}$. Dirac singularities are located at the discontinuity points of the 2π periodic function $\daleth(\gamma)$. Actually, Pauli theorem and its correct forms prevent the corresponding quantum commutator from being exactly canonical.

4. Affine quantization

References

[1] H Bergeron, A Dapor, J-P G and P Małkiewicz, *Wavelet Quantum Cosmology* (2013); arXiv:1305.0653 [gr-qc]

[2] H Bergeron, A Dapor, J-P G and P Małkiewicz, *Towards singularity-free cosmology: coherent state quantization* (2013)

Affine or Wavelet Quantization

- Set X is the upper half-plane $\Pi_+ := \{(q,p)\,|\,p \in \mathbb{R}\,,\ q > 0\}$ equipped with measure $\mathrm{d}q\,\mathrm{d}p$. It is the phase space for the motion on the half-line.
- Equipped with the multiplication $(q,p)(q_0,p_0) = (qq_0, p_0/q + p)$, $q \in \mathbb{R}^*_+$, $p \in \mathbb{R}$, X is viewed as the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line.
- $\mathrm{Aff}_+(\mathbb{R})$ has two non-equivalent UIR, $U_\pm$. Both are square integrable $\Rightarrow$ *continuous wavelet analysis*.
- $U_+ \equiv U$ carried on by Hilbert space $\mathcal{H} = L^2(\mathbb{R}^*_+, dx)$:

$$U(q,p)\psi(x) = (e^{ipx}/\sqrt{q})\psi(x/q)\,.$$

- unit-norm state $\psi \in L^2(\mathbb{R}^\dagger_+,\ \mathrm{d}x) \cap L^2(\mathbb{R}^\dagger_+,\ \mathrm{d}x/x)$ (“fiducial vector”) produces all wavelet $\Leftrightarrow$ CS defined as $|q,p\rangle = U(q,p)|\psi\rangle$ and yielding the crucial

$$\int_{\Pi_+} \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi c_{-1}} |q,p\rangle\langle q,p| = I\,,\ c_\gamma := \int_0^\infty \mathrm{d}x|\psi(x)|^2/x^{2+\gamma}\,.$$

Wavelet Quantization continued

- Covariant quantization from resolution of the identity[a]

$$f \mapsto A_f = \int_{\Pi_+} \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi c_{-1}} f(q,p) |q,p\rangle\langle q,p|$$

- Quantization is canonical (up to a multiplicative constant) for q and p:

$$A_p = P = -i\partial/\partial x\,,\; A_{q^\beta} = (c_{\beta-1}/c_{-1})\, Q^\beta\,,\; Qf(x) = xf(x)\,,$$

- Quantization of kinetic energy:

$$A_{p^2} = P^2 + KQ^{-2}\,, \quad K = K(\psi) = \int_0^\infty u\,\mathrm{d}u (\psi'(u))^2/c_{-1}$$

  Thus wavelet quantization forbids a quantum free particle moving on the positive line to reach the origin.

- Operator $P^2 = -d^2/dx^2$ alone in $L^2(\mathbb{R}_+^*, \mathrm{d}x)$ is not essentially self-adjoint whereas the above regularized operator, defined on the domain of smooth compactly supported functions, is for $K \geqslant 3/4$[b]. Then quantum dynamics of the free motion is possible.

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[a] Proceeding in quantum theory with an "affine" quantization instead of the Weyl-Heisenberg quantization was already present in Klauder's work devoted the question of dealing with singularities in quantum gravity (see An Affinity for Affine Quantum Gravity, *Proc. Steklov Inst. of Math.* **272**, 169-176 (2011); gr-qc/1003.261 for recent references). The procedure rests on the representation of the affine Lie algebra. In this sense, it remains closer to the canonical one and it is not of the integral type.

[b] F. Gesztesy and W. Kirsch J. Rein. Ang. Math. **362** 28 (1985)

Semi-classical aspects in phase space

- Quantum states and their dynamics have phase space representation through wavelet symbols. For state $|\phi\rangle$:

$$\Phi(q,p) = \langle q,p|\phi\rangle/\sqrt{2\pi}$$

- Associated probability distribution on phase space:

$$\rho_\phi(q,p) = \frac{1}{2\pi c_{-1}}|\langle q,p|\phi\rangle|^2$$

- With (energy) eigenstates of some quantum Hamiltonian H at our disposal, we can compute the time evolution

$$\rho_\phi(q,p,t) := \frac{1}{2\pi c_{-1}}|\langle q,p|e^{-iH}|\phi\rangle|^2$$

for any state ϕ.

Wavelet Quantization for FLRW Quantum Cosmology

- FLRW models filled with barotropic fluid with equation of state $\mathrm{p} = w\rho$ and resolving Hamiltonian constraint leads to a model of singular universe $\sim$ particle moving on the half-line $(0, \infty)$ with Hamiltonian.

$$\{q, p\} = 1, \;\; h(q, p) = \alpha(w)p^2 + 6\tilde{k}q^{\beta(w)}, \;\; q > 0\,.$$

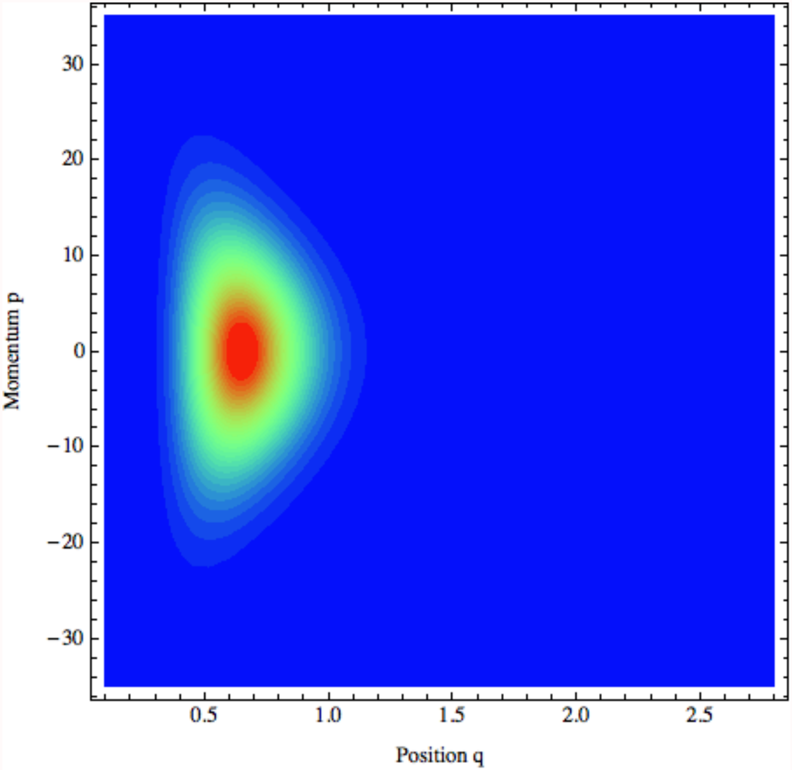
with $\tilde{k} = (\int \mathrm{d}\omega)^{2/3}k$, $\alpha(w) = 3(1-w)^2/32$ and $\beta(w) = 2(3w+1)/(3(1-w))$. $k = 0, -1$ or 1 (in suitable unit of inverse area) depending on whether the universe is flat, open or closed.

- Assume a closed universe with radiation content : $w = 1/3$ and $k = +1$. Affine quantization with a fiducial vector like $\psi(x) \propto \exp(-(\alpha(\nu)x + \beta(\nu)/x)$, whith parameter $\nu > 0$, on $\mathbb{R}_+^*$ yields the quantum Hamiltonian

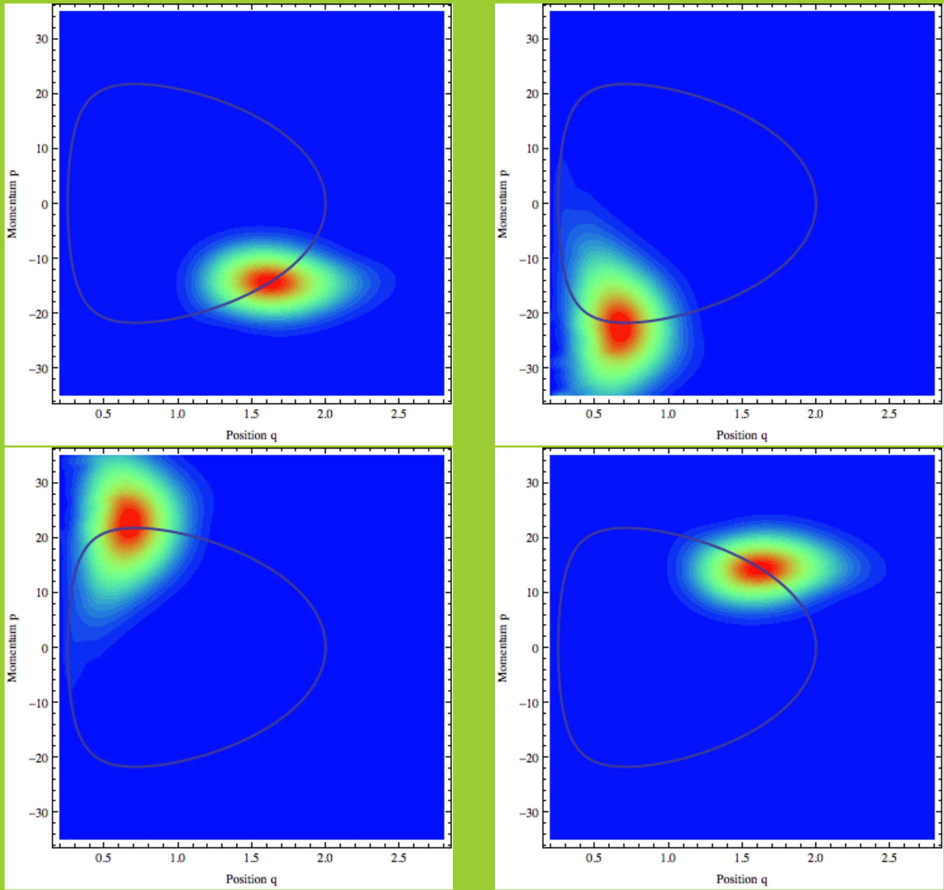
$$A_h = H = \frac{1}{24}P^2 + \frac{a_P^2 K(\nu)}{24}\frac{1}{Q^2} + 6\frac{a_P^2}{\sigma^2}\frac{c_1}{c_{-1}}Q^2\,,$$

a_P is a Planck area.

- For $K(\nu) \geqslant 3/4$ wavelet quantization removes quantum singularity and well-defined quantum evolution exists, at the difference with canonical quantization



Phase space distribution of the ground state with a certain choice of ν. $a_P = 1$. This stationary quantum state of the universe is distributed around the equilibrium point q_e (minimum of the potential curve involved in the Hamiltonian). The existence of the semi-classical equilibrium point $q_e \neq 0$ is a consequence of the repulsive part of the potential.



Phase space distribution $\rho_{q_0,p_0,t}(q,p)$ for some selected values of time t. (Fluid configuration variable is chosen as a clock of universe). The thick curve is the phase trajectory obtained from the effective dynamics.

A “semiclassical” Friedmann equation

- In general lower symbol $\check{f}(q,p)$ differs from its classical counterpart $f(q,p)$: it is a quantum-corrected effective observable.
- Thus, computing lower symbol of Hamiltonian leads to the semiclassical Friedmann equation for scale factor $a(t)$:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} + c^2 a_P^2 (1-w)^2 \frac{\nu}{128} \frac{1}{V^2} = \frac{8\pi G}{3c^2}\rho$$

- Note that the repulsive potential depends explicitly on volume. This excludes non-compact universes from quantum modeling.
- Singularity resolution is confirmed: as the singular geometry is approached ($a \to 0$), the repulsive potential grows faster ($\sim a^{-6}$) than the density of fluid ($\sim a^{-3(1+w)}$) and therefore at some point the two terms become equal and the contraction is brought to a halt.
- **The form of the repulsive potential does not depend on the state of fluid filling the universe: the origin of singularity avoidance is quantum geometrical.**

5. Conclusion

Beyond the freedom (think to analogy with Signal Analysis where different techniques are complementary) allowed by integral quantization, the advantages of the method with regard to other quantization procedures in use are of four types.

(i) The minimal amount of constraints imposed to the classical objects to be quantized.

(ii) Once a choice of (positive) operator-valued measure has been made, which must be consistent with experiment, there is no ambiguity in the issue, contrarily to other method(s) in use (think in particular to the ordering problem). To one classical object corresponds one and only one quantum object. Of course different choices are requested to be physically equivalent

(iii) The method produces in essence a regularizing effect, at the exception of certain choices, like the Weyl-Wigner integral quantization.

(iv) The method, through POVM choices, offers the possibility to keep a full probabilistic content. As a matter of fact, the Weyl-Wigner integral quantization does not rest on a POVM.

- But what is the real meaning of that freedom granted to us in the choice of POVM or others?
- Such a freedom is governed by our degree of confidence in localizing a pure classical state (q, p) in phase space. The latter is usually viewed as an ideal continuous manifold where all points are physically accessible. As everybody knows, such a view is physically untenable ...
- However, and this is the paradoxical paradigm of contemporary physics, one needs such a leibnizian mathematical ideality (*natura non saltum facit*) to build a more realistic, though more highly mathematical, representation of the physical world.