Universality in bootstrap percolation and kinetically constrained models[1]

Ivailo Hartarsky
CEREMADE, Université Paris-Dauphine, PSL University

15 June 2022

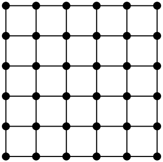
Colóquio Interinstitutional Modelos Estocásticos e Aplicações, Rio de Janeiro

[1] Supported by ERC Starting Grant 680275 MALIG

Bootstrap percolation

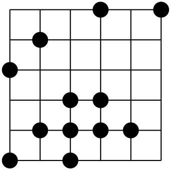
Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.



Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).



Bootstrap percolation

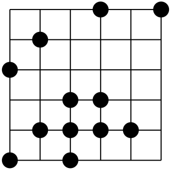
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet =$healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.

Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
    - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.

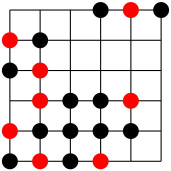
Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



Bootstrap percolation

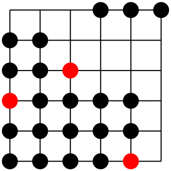
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

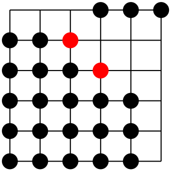
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

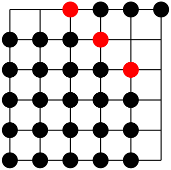
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

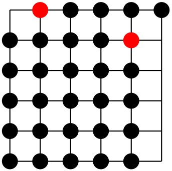
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

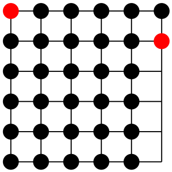
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

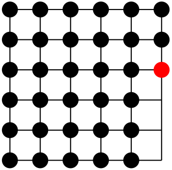
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t + 1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

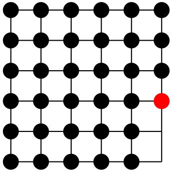
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

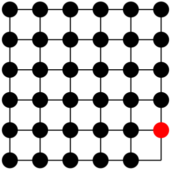
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

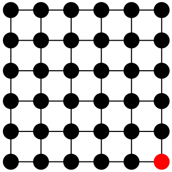
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t + 1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

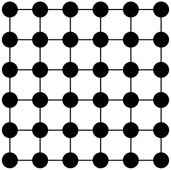
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

Bootstrap percolation

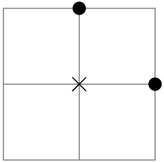
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Example: j-neighbour update family, $j \in \{1, \dots, 4\}$.
  - If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least j neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
  - $\bullet$ never becomes $\circ$.



$j = 2$

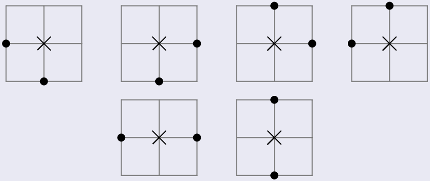
Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.



Bootstrap percolation

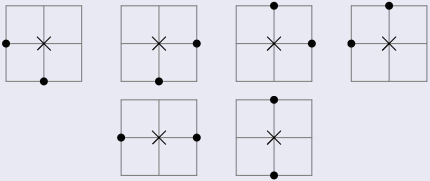
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.



Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet .$$

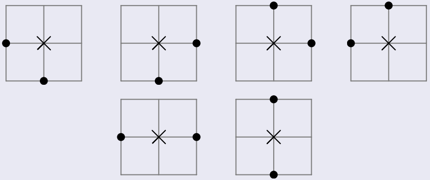


Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet .$$

2-neighbour bootstrap percolation



Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet .$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.

Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet .$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.

Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet.$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.
- Initial distribution: $\pi = Ber(p)^{\otimes \mathbb{Z}^2}$.

Bootstrap percolation

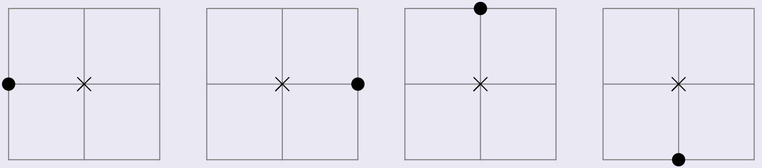
- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet.$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.
- Initial distribution: $\pi = Ber(p)^{\otimes \mathbb{Z}^2}$.
- Critical probability: $p_c = \inf\{p \in [0, 1] : \pi(\tau_0 = \infty) = 0\}$.

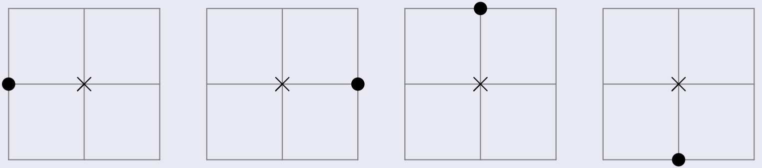
Examples

1-neighbour



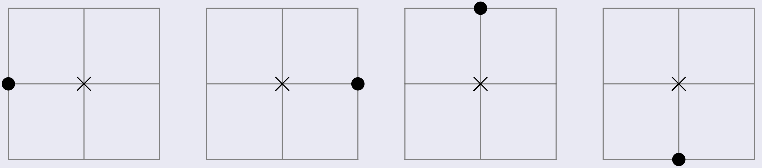
Examples

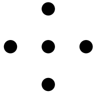
1-neighbour



Examples

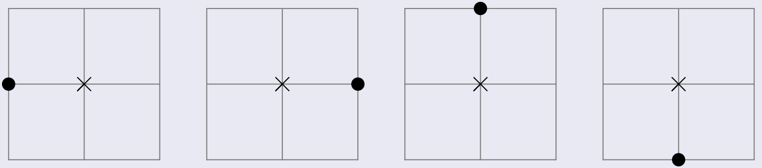
1-neighbour

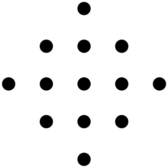




Examples

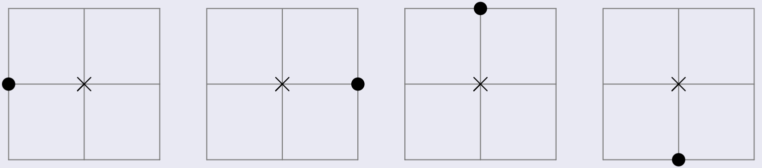
1-neighbour

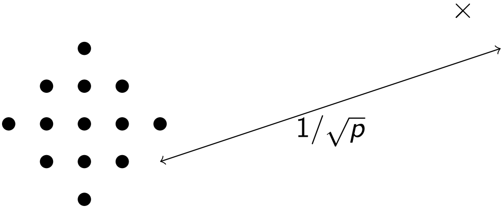




Examples

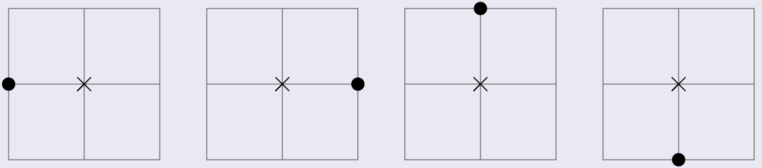
1-neighbour





Examples

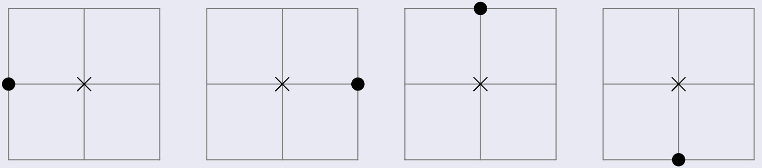
1-neighbour



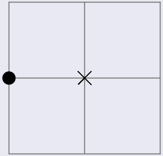
$p_{\mathrm{c}} = 0$ $\tau_0 \approx 1/\sqrt{p}$

Examples

1-neighbour

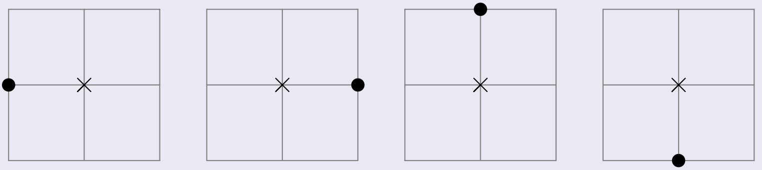


East

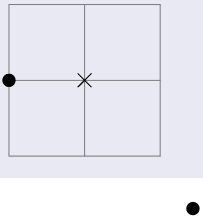


Examples

1-neighbour

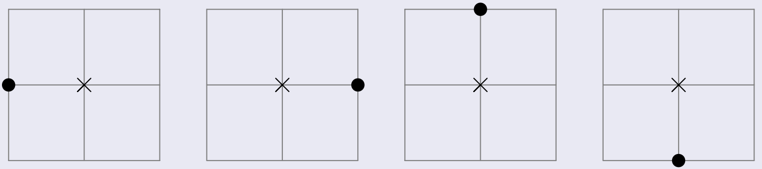


East

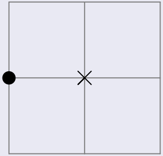


Examples

1-neighbour

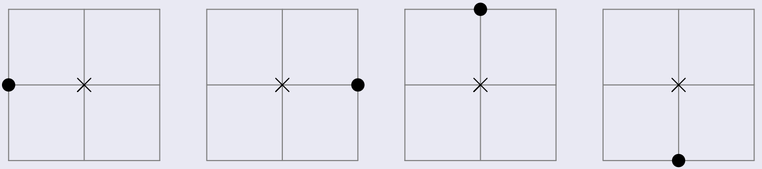


East

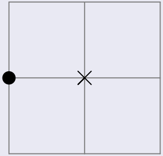


Examples

1-neighbour

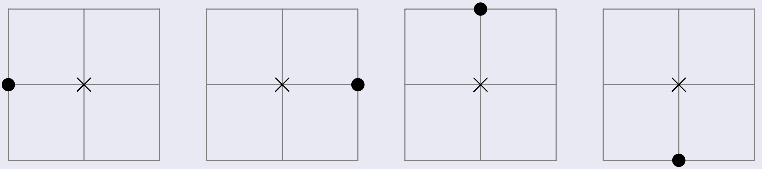


East

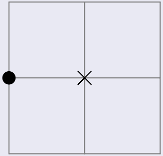


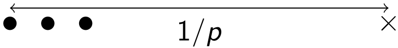
Examples

1-neighbour



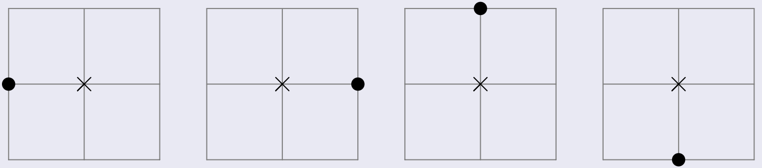
East



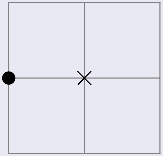


Examples

1-neighbour



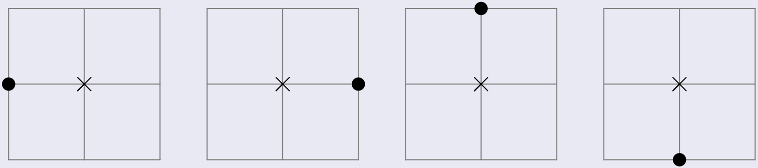
East



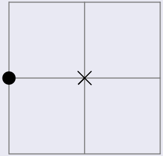
$p_c = 0$ $\tau_0 \approx 1/p$

Examples

1-neighbour



East



$$\tau_0 \sim \mathcal{G}(p)$$

Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

Theorem (Bollobás–Smith–Uzzell'15)

If $\mathcal{U}$ is supercritical, then $p_c = 0$ and $\tau_0 = p^{-\Theta(1)}$.

Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

Theorem (Bollobás–Smith–Uzzell'15)

If $\mathcal{U}$ is supercritical, then $p_{\mathrm{c}} = 0$ and $\tau_0 = p^{-\Theta(1)}$.

Definition (Stable directions)

A direction $u \in S^1$ is *unstable* if there exists $U \in \mathcal{U}$ contained in

$$U \subset \mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}.$$

Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

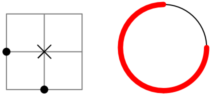
Theorem (Bollobás–Smith–Uzzell'15)

If $\mathcal{U}$ is supercritical, then $p_{\mathrm{c}} = 0$ and $\tau_0 = p^{-\Theta(1)}$.

Definition (Stable directions)

A direction $u \in S^1$ is *unstable* if there exists $U \in \mathcal{U}$ contained in

$$U \subset \mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}.$$



Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

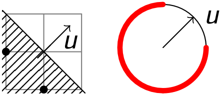
Theorem (Bollobás–Smith–Uzzell'15)

If $\mathcal{U}$ is supercritical, then $p_c = 0$ and $\tau_0 = p^{-\Theta(1)}$.

Definition (Stable directions)

A direction $u \in S^1$ is *unstable* if there exists $U \in \mathcal{U}$ contained in

$$U \subset \mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}.$$



Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

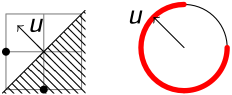
Theorem (Bollobás–Smith–Uzzell'15)

If $\mathcal{U}$ is supercritical, then $p_c = 0$ and $\tau_0 = p^{-\Theta(1)}$.

Definition (Stable directions)

A direction $u \in S^1$ is *unstable* if there exists $U \in \mathcal{U}$ contained in

$$U \subset \mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}.$$



Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

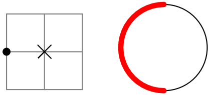
Theorem (Bollobás–Smith–Uzzell'15)

If $\mathcal{U}$ is supercritical, then $p_c = 0$ and $\tau_0 = p^{-\Theta(1)}$.

Definition (Stable directions)

A direction $u \in S^1$ is *unstable* if there exists $U \in \mathcal{U}$ contained in

$$U \subset \mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}.$$



Definition (Supercritical family)

An update family $\mathcal{U}$ is *supercritical* if a finite set $Z \subset \mathbb{Z}^2$ of infections can infect an infinite one.

Theorem (Bollobás–Smith–Uzzell'15)

If $\mathcal{U}$ is supercritical, then $p_{\mathrm{c}} = 0$ and $\tau_0 = p^{-\Theta(1)}$.

Definition (Stable directions)

A direction $u \in S^1$ is *unstable* if there exists $U \in \mathcal{U}$ contained in

$$U \subset \mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}.$$

Theorem (BSU15)

An update family $\mathcal{U}$ is supercritical iff there is an open semi-circle of unstable directions.

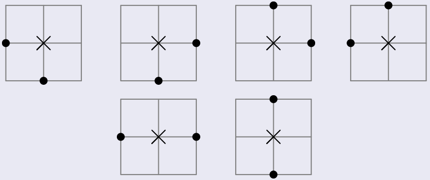
Definition (Critical family)

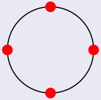
An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

2-neighbour bootstrap percolation





Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

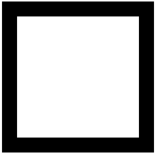
If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ *is critical, then* $p_\mathrm{c} = 0$ *and* $\tau_0 = \exp(p^{-\Theta(1)})$.

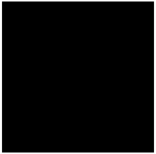


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

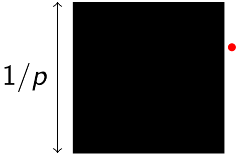


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

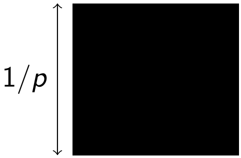


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ *is critical, then* $p_{\mathrm{c}} = 0$ *and* $\tau_0 = \exp(p^{-\Theta(1)})$.

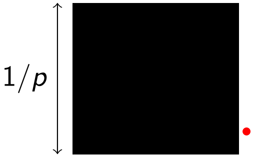


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

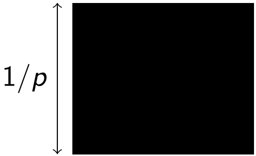


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

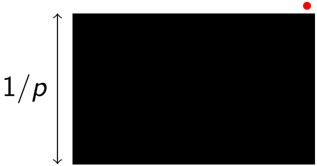


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_{\mathrm{c}} = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

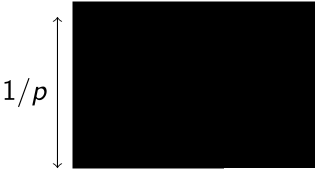


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

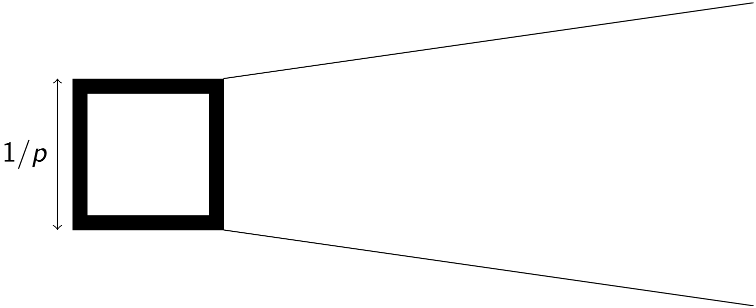


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ *is critical, then* $p_{\mathrm{c}} = 0$ *and* $\tau_0 = \exp(p^{-\Theta(1)})$.

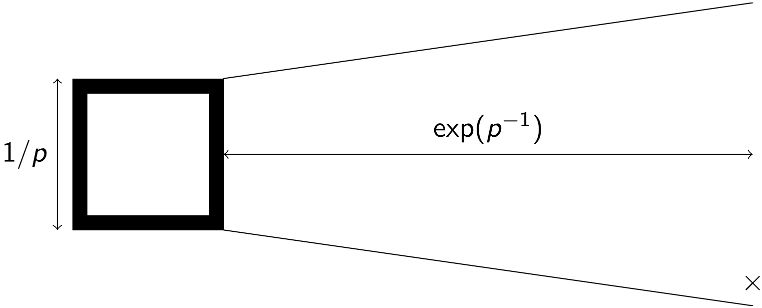


Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.



Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_{\mathrm{c}} = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

Definition (Difficulty)

The *difficulty* $\alpha(u) \in \{1, 2, \dots\}$ of an isolated stable direction $u \in S^1$ is the smallest cardinal of a set of $Z \subset \mathbb{Z}^2$ such that $Z \cup \mathbb{H}_u$ can infect an infinite set. We set $\alpha(u) = \infty$ for non-isolated stable directions and $\alpha(u) = 0$ for unstable ones.

Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

Definition (Difficulty)

The *difficulty* $\alpha(u) \in \{1, 2, \dots\}$ of an isolated stable direction $u \in S^1$ is the smallest cardinal of a set of $Z \subset \mathbb{Z}^2$ such that $Z \cup \mathbb{H}_u$ can infect an infinite set. We set $\alpha(u) = \infty$ for non-isolated stable directions and $\alpha(u) = 0$ for unstable ones. The difficulty of $\mathcal{U}$ is $\alpha = \alpha(\mathcal{U}) = \min_C \max_{u \in C} \alpha(u)$, where the minimum is over open semi-circles.

Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

Theorem (BSU15, Bollobás–Duminil-Copin–Morris–Smith'14+)

If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\alpha+o(1)})$.

Definition (Difficulty)

The *difficulty* $\alpha(u) \in \{1, 2, \dots\}$ of an isolated stable direction $u \in S^1$ is the smallest cardinal of a set of $Z \subset \mathbb{Z}^2$ such that $Z \cup \mathbb{H}_u$ can infect an infinite set. We set $\alpha(u) = \infty$ for non-isolated stable directions and $\alpha(u) = 0$ for unstable ones. The difficulty of $\mathcal{U}$ is $\alpha = \alpha(\mathcal{U}) = \min_C \max_{u \in C} \alpha(u)$, where the minimum is over open semi-circles.

Definition (Critical family)

An update family is *critical* if there is no unstable open semi-circle, but there exists a semi-circle with finitely many stable directions.

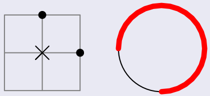
Theorem (H–Mezei'20)

Given a critical update family $\mathcal{U}$, determining its difficulty $\alpha(\mathcal{U})$ is NP-hard, but computable.

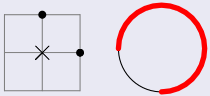
Definition (Difficulty)

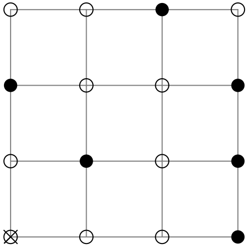
The *difficulty* $\alpha(u) \in \{1, 2, \dots\}$ of an isolated stable direction $u \in S^1$ is the smallest cardinal of a set of $Z \subset \mathbb{Z}^2$ such that $Z \cup \mathbb{H}_u$ can infect an infinite set. We set $\alpha(u) = \infty$ for non-isolated stable directions and $\alpha(u) = 0$ for unstable ones. The difficulty of $\mathcal{U}$ is $\alpha = \alpha(\mathcal{U}) = \min_C \max_{u \in C} \alpha(u)$, where the minimum is over open semi-circles.

North-East/Oriented percolation

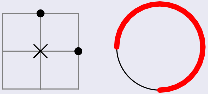


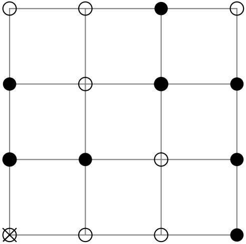
North-East/Oriented percolation



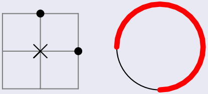


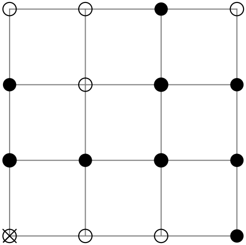
North-East/Oriented percolation



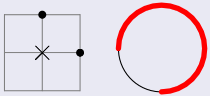


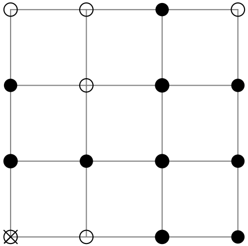
North-East/Oriented percolation



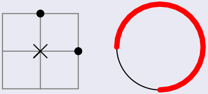


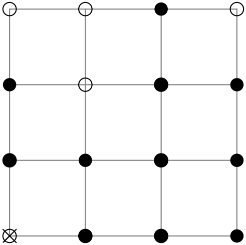
North-East/Oriented percolation



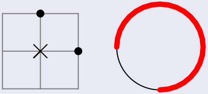


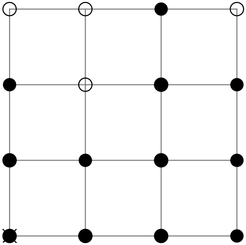
North-East/Oriented percolation





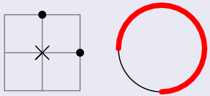
North-East/Oriented percolation

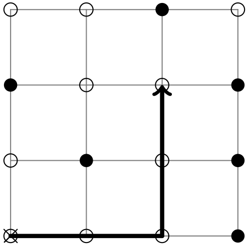




$$\tau_0 = 5$$

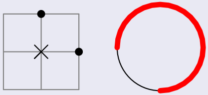
North-East/Oriented percolation





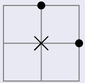
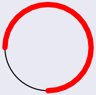
$\tau_0 = 5$

North-East/Oriented percolation



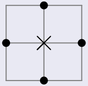
$$p_c \in (0, 1)$$

North-East/Oriented percolation

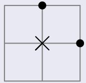



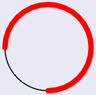
$$p_c \in (0,1)$$

4-neighbour bootstrap percolation



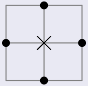

North-East/Oriented percolation





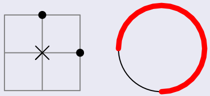
$$p_c \in (0,1)$$

4-neighbour bootstrap percolation



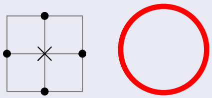
$$p_c = 1$$

North-East/Oriented percolation

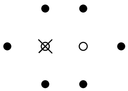


$$p_c \in (0,1)$$

4-neighbour bootstrap percolation



$$p_c = 1$$



Definition (Subcritical family)

An update family is *subcritical* if every semi-circle contains infinitely many stable directions. It is *trivial subcritical* if all directions are stable.

Definition (Subcritical family)

An update family is *subcritical* if every semi-circle contains infinitely many stable directions. It is *trivial subcritical* if all directions are stable.

Theorem (Balister–Bollobás–Przykucki–Smith'16)

If $\mathcal{U}$ is subcritical, then $p_{\mathrm{c}} > 0$. Moreover, $p_{\mathrm{c}} = 1$ iff it is trivial subcritical.

Definition (Subcritical family)

An update family is *subcritical* if every semi-circle contains infinitely many stable directions. It is *trivial subcritical* if all directions are stable.

Theorem (Balister–Bollobás–Przykucki–Smith'16)

If $\mathcal{U}$ is subcritical, then $p_c > 0$. Moreover, $p_c = 1$ iff it is trivial subcritical.

Conjecture (Schonmann'92,H'21)

For all $\mathcal{U}$ and $p > p_c$, τ_0 has an exponential moment.

Definition (Subcritical family)

An update family is *subcritical* if every semi-circle contains infinitely many stable directions. It is *trivial subcritical* if all directions are stable.

Theorem (Balister–Bollobás–Przykucki–Smith'16)

If $\mathcal{U}$ is subcritical, then $p_{\mathrm{c}} > 0$. Moreover, $p_{\mathrm{c}} = 1$ iff it is trivial subcritical.

Conjecture (Schonmann'92,H'21)

For all $\mathcal{U}$ and $p > p_{\mathrm{c}}$, τ_0 has an exponential moment.

Theorem (H'22)

For all $\mathcal{U}$ supported in a half-space the conjecture holds.

Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet .$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.
- Initial distribution: $\pi = Ber(p)^{\otimes \mathbb{Z}^2}$.
- Critical probability: $p_c = \inf\{p \in [0, 1] : \pi(\tau_0 = \infty) = 0\}$.

Kinetically constrained models

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-KCM infections can heal and at rate 1 we update to $Ber(p)$ all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet .$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{R}_+ : 0 \text{ is } \bullet\} \in \mathbb{R}_+ \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.
- Initial and stationary distribution: $\pi = Ber(p)^{\otimes \mathbb{Z}^2}$.
- Critical probability: $p_c = \inf\{p \in [0, 1] : \mathbb{P}_\pi(\tau_0 = \infty) = 0\}$.

Theorem (Cancrini–Martinelli–Roberto–Toninelli'08)

For any $\mathcal{U}$ the following are equivalent:

- $\pi(\tau_0 = \infty) = 0$ *in $\mathcal{U}$-bootstrap percolation;*
- $\mathbb{P}_\pi(\tau_0 = \infty) = 0$ *in the $\mathcal{U}$-KCM;*
- 0 *is a simple eigenvalue of the generator of the $\mathcal{U}$-KCM;*
- *the $\mathcal{U}$-KCM is ergodic;*
- *the $\mathcal{U}$-KCM is mixing.*

Theorem (Cancrini–Martinelli–Roberto–Toninelli'08)

For any $\mathcal{U}$ the following are equivalent:

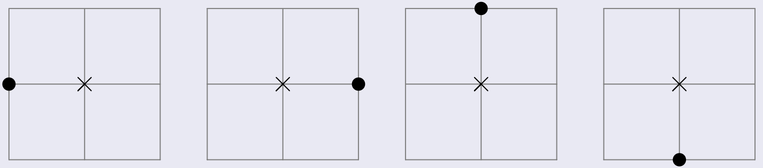
- $\pi(\tau_0 = \infty) = 0$ *in $\mathcal{U}$-bootstrap percolation;*
- $\mathbb{P}_\pi(\tau_0 = \infty) = 0$ *in the $\mathcal{U}$-KCM;*
- 0 *is a simple eigenvalue of the generator of the $\mathcal{U}$-KCM;*
- *the $\mathcal{U}$-KCM is ergodic;*
- *the $\mathcal{U}$-KCM is mixing.*

Theorem (CMRT08,H21)

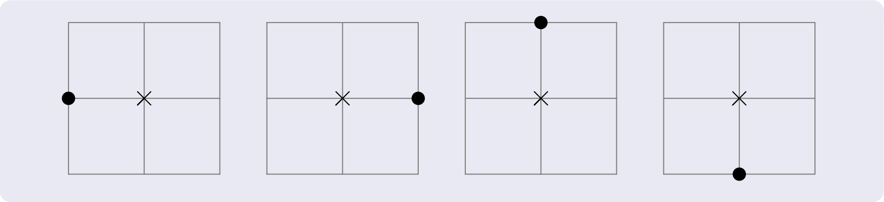
For any $\mathcal{U}$ the following are equivalent:

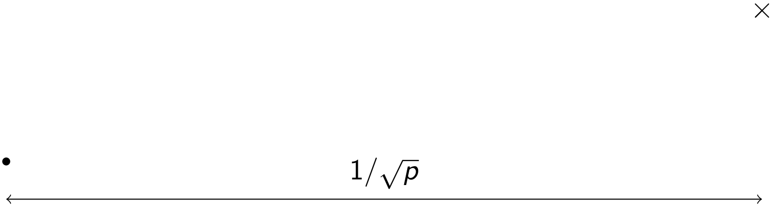
- *in $\mathcal{U}$-bootstrap percolation τ_0 has an exponential moment;*
- *in $\mathcal{U}$-KCM τ_0 has an exponential moment;*
- $T_{\mathrm{rel}} < \infty$ *for the $\mathcal{U}$-KCM.*

1-neighbour KCM

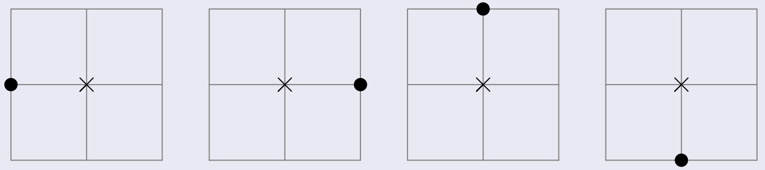


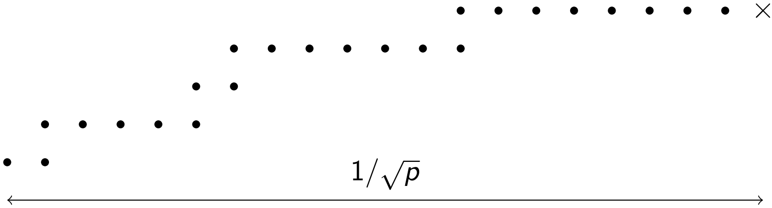
1-neighbour KCM



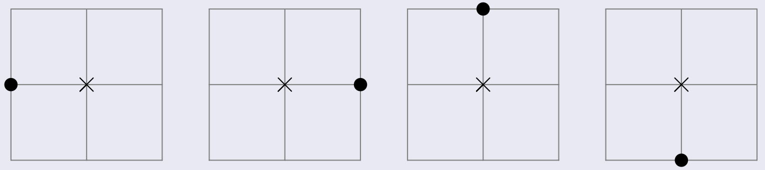


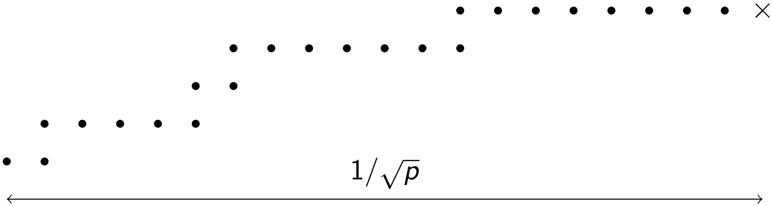
1-neighbour KCM





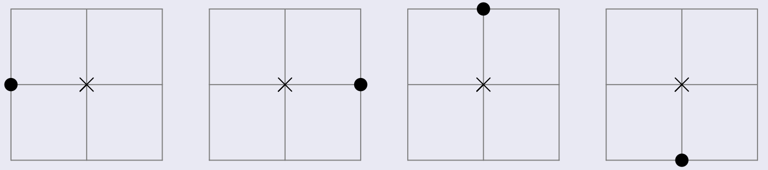
1-neighbour KCM

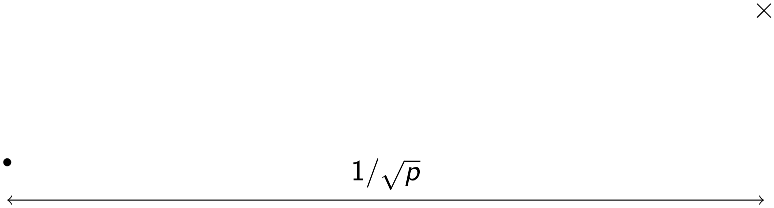




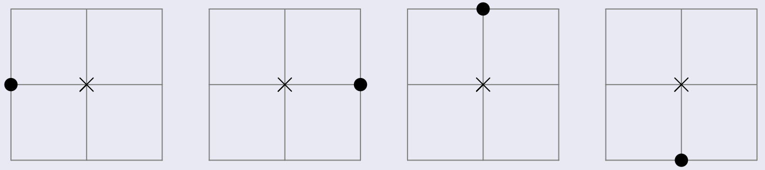
$$\tau_0 \leqslant \exp(1/\sqrt{p})$$

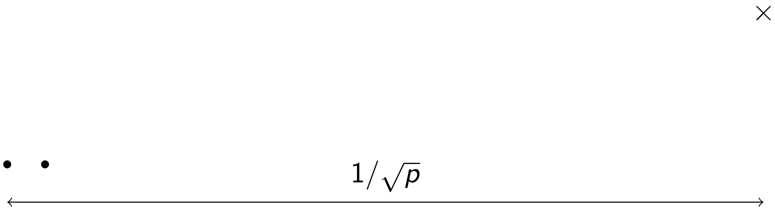
1-neighbour KCM



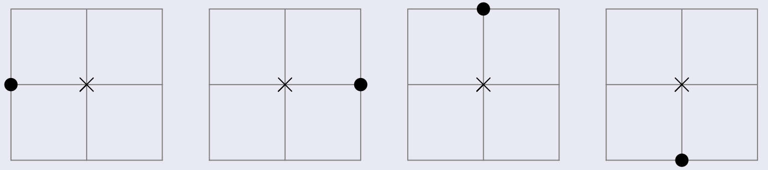


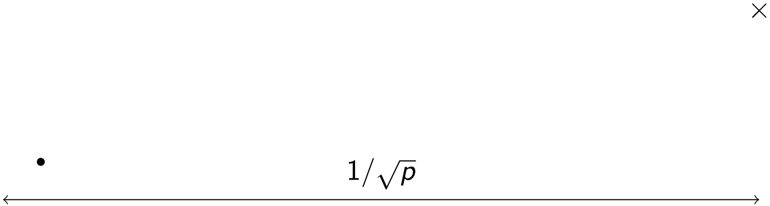
1-neighbour KCM



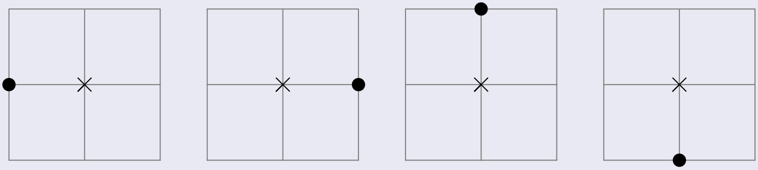


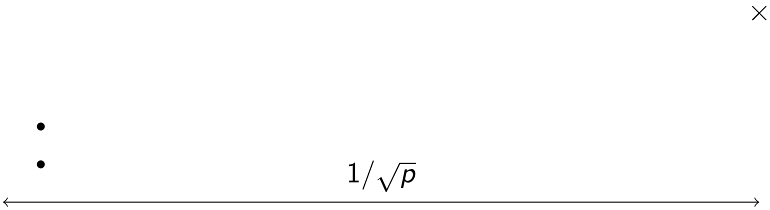
1-neighbour KCM



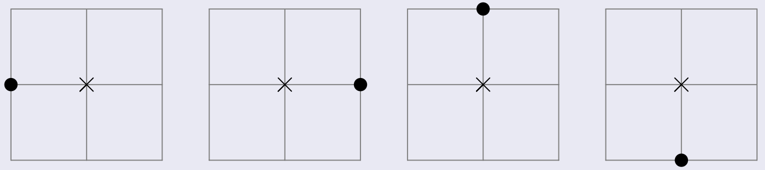


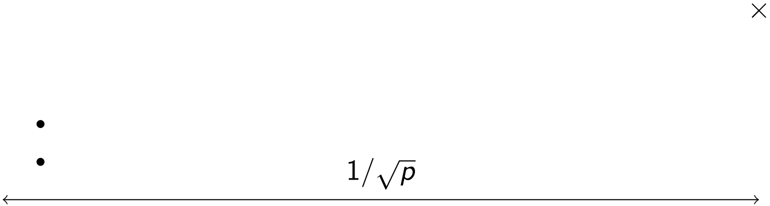
1-neighbour KCM





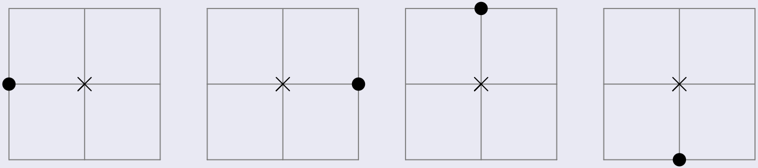
1-neighbour KCM





$$\tau_0 \leqslant (1/\sqrt{p})^2/p = 1/p^2$$

1-neighbour KCM

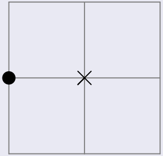


Theorem (CMRT08,Shapira'20)

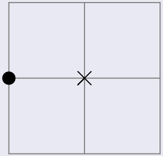
For the 1-neighbour KCM we have

$$T_{\mathrm{rel}} = \begin{cases} \Theta(p^{-3}) & d = 1, \\ p^{-2+o(1)} & d = 2, \\ \Theta(p^{-2}) & d \geqslant 3. \end{cases}$$

East



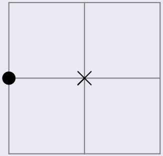
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

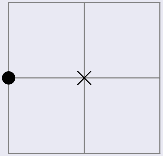
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

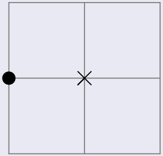
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1}-1$.

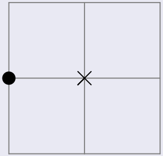
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

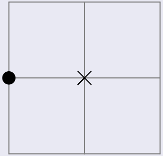
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

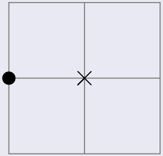
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

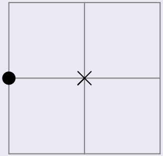
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

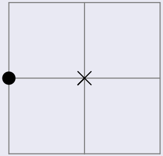
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

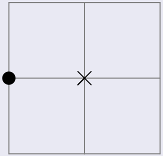
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

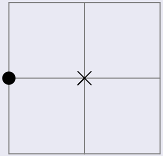
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

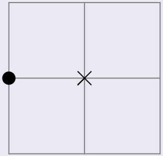
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

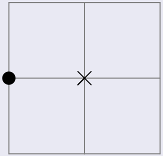
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1}-1$.

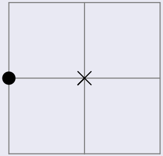
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

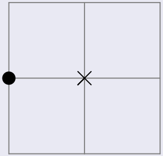
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

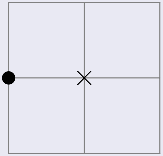
East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1}-1$.

East



Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1}-1$.

Theorem (Aldous–Diaconis'02, CMRT08)

$$T_{\mathrm{rel}} = \exp\left(\frac{\log^2(1/p)}{2\log 2 + o(1)}\right).$$

Supercritical KCM

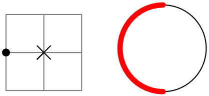
Definition (Rooted)

A supercritical family $\mathcal{U}$ is *rooted* if there exist two non-opposite stable directions and *unrooted* otherwise.

Supercritical KCM

Definition (Rooted)

A supercritical family $\mathcal{U}$ is *rooted* if there exist two non-opposite stable directions and *unrooted* otherwise.



Supercritical KCM

Definition (Rooted)

A supercritical family $\mathcal{U}$ is *rooted* if there exist two non-opposite stable directions and *unrooted* otherwise.

Theorem (Martinelli–Toninelli'19, Martinelli–Morris–Toninelli'19, Marêché'20, Marêché–Martinelli–Toninelli'20)

For a supercritical KCM we have

- $T_{\mathrm{rel}} = p^{-\Theta(1)}$ *if* $\mathcal{U}$ *is unrooted;*
- $T_{\mathrm{rel}} = \exp(\Theta(\log^2(1/p)))$ *if* $\mathcal{U}$ *is rooted.*

Critical KCM

Theorem (MT19, MMT19)

For critical $\mathcal{U}$-KCM we have $\tau_0 = \exp(p^{-\Theta(1)})$.

Critical KCM

Theorem (MT19, MMT19)

For critical $\mathcal{U}$-KCM we have $\tau_0 = \exp(p^{-\Theta(1)})$.

Conjecture (MMT19)

The correct exponent is between α and 2α. It corresponds to either moving 1-neighbour-like or East-like (whichever is more efficient).

Critical KCM

Theorem (MT19, MMT19)

For critical $\mathcal{U}$-KCM we have $\tau_0 = \exp(p^{-\Theta(1)})$.

Conjecture (MMT19)

The correct exponent is between α and 2α. It corresponds to either moving 1-neighbour-like or East-like (whichever is more efficient).

Theorem (MMT19, H–Marêché–Toninelli'20)

For a critical $\mathcal{U}$-KCM with infinite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-2\alpha+o(1)})$.

Critical KCM

Theorem (MT19, MMT19)

For critical $\mathcal{U}$-KCM we have $\tau_0 = \exp(p^{-\Theta(1)})$.

Conjecture (MMT19)

The correct exponent is between α and 2α. It corresponds to either moving 1-neighbour-like or East-like (whichever is more efficient).

Theorem (MMT19, H–Marêché–Toninelli'20)

For a critical $\mathcal{U}$-KCM with infinite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-2\alpha+o(1)})$.

- East-like movement of 'droplets'.

Critical KCM

Theorem (MT19, MMT19)

For critical $\mathcal{U}$-KCM we have $\tau_0 = \exp(p^{-\Theta(1)})$.

Conjecture (MMT19)

The correct exponent is between α and 2α. It corresponds to either moving 1-neighbour-like or East-like (whichever is more efficient).

Theorem (MMT19, H–Marêché–Toninelli'20)

For a critical $\mathcal{U}$-KCM with infinite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-2\alpha+o(1)})$.

- East-like movement of 'droplets'.
- East-like deterministic logarithmic bottleneck.

Conjecture (MMT19)

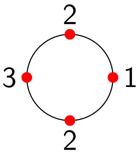
The correct exponent is between α and 2α. It corresponds to either moving 1-neighbour-like or East-like (whichever is more efficient).

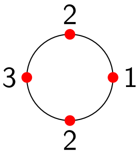
Theorem (MMT19, HMT20)

For a critical $\mathcal{U}$-KCM with infinite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-2\alpha+o(1)})$.

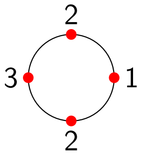
Theorem (H–Martinelli–Toninelli'21)

For a critical $\mathcal{U}$-KCM with finite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-\alpha+o(1)})$.

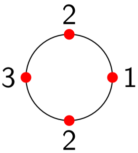

2
3
1
2



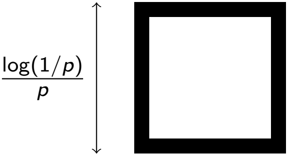
$$\alpha = 1$$

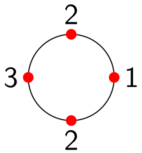


$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$

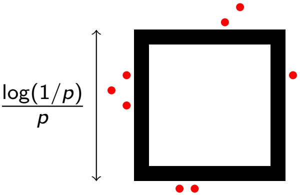


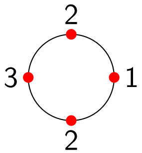
$$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$$



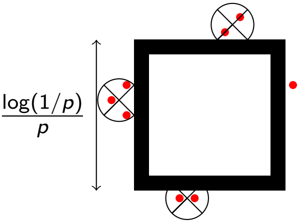


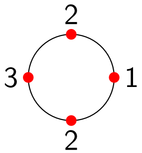
$$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$$



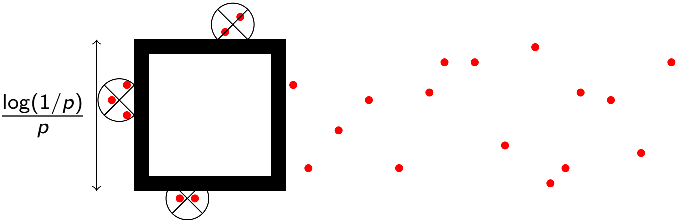


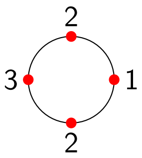
$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$



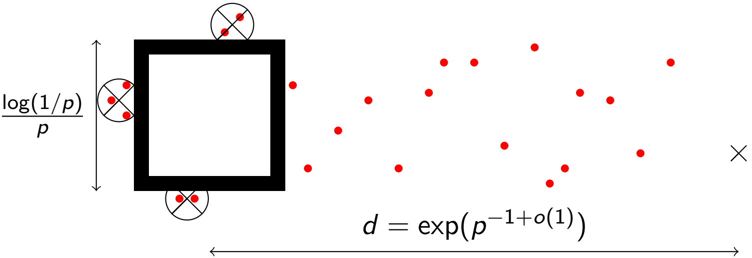


$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$

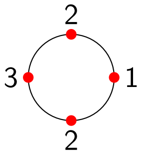




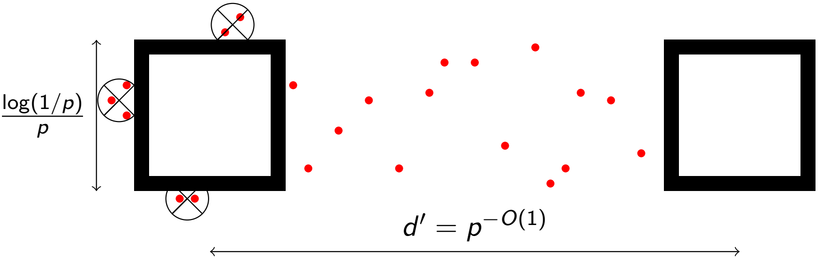
$\alpha = 1 \qquad \tau_0 \leqslant \exp(p^{-1+o(1)})$

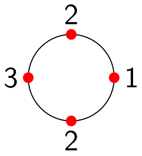


$$\tau_0 \leqslant (\exp(p^{-1+o(1)}))^{\log d} = \exp(p^{-2+o(1)})$$

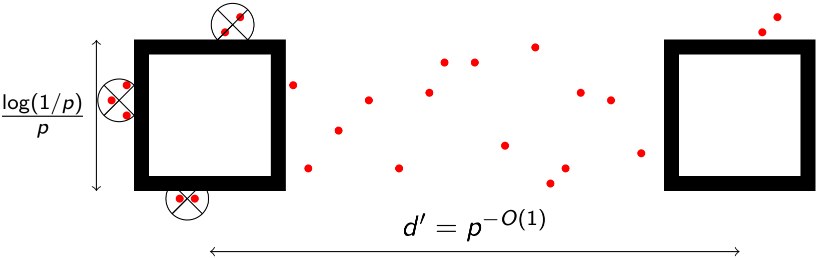


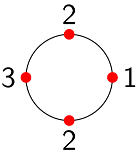
$\alpha = 1 \qquad \tau_0 \leqslant \exp(p^{-1+o(1)})$



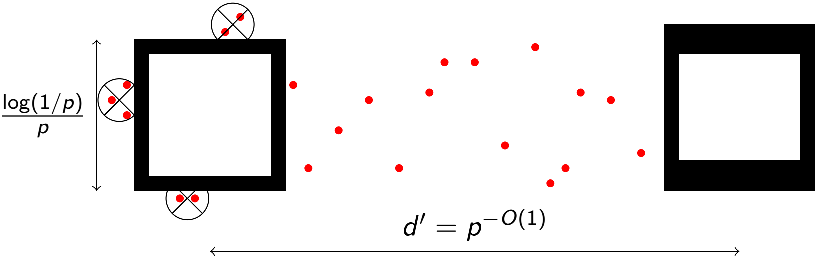


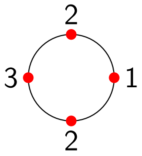
$\alpha = 1 \qquad \tau_0 \leqslant \exp(p^{-1+o(1)})$



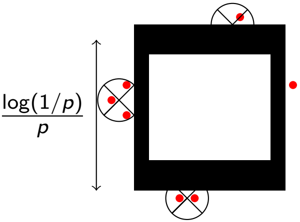


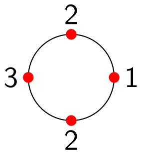
$$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$$



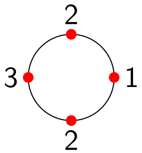


$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$

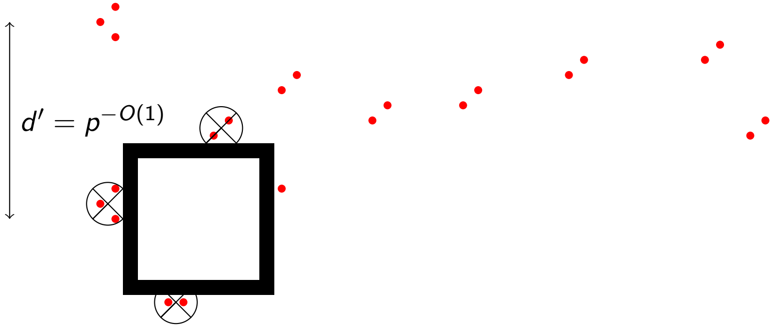


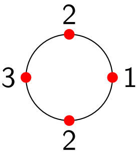


$\alpha = 1 \qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$

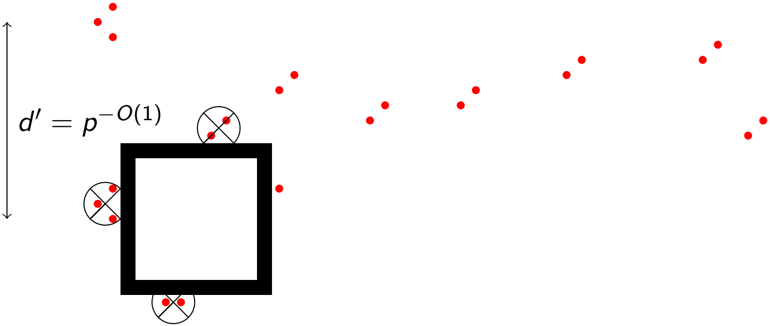


$\alpha = 1 \qquad \tau_0 \leqslant \exp(p^{-1+o(1)})$





$\alpha = 1$ $\qquad \tau_0 \leqslant \exp\left(p^{-1+o(1)}\right)$



$$\tau_0 \leqslant \left(\exp\left(p^{-1+o(1)}\right)\right)^{\log d'} = \exp\left(p^{-1+o(1)}\right)$$

Refined universality

Definition

Fix a critical update family. A direction u is *hard* if $\alpha(u) > \alpha$. The family is *unbalanced* if there are two opposite hard directions and *balanced* otherwise.

Theorem (BDCMS14+)

For critical $\mathcal{U}$-bootstrap percolation with difficulty α we have

$$\tau_0 = \exp\left(\Theta(1)\frac{1}{p^\alpha}\left(\log\frac{1}{p}\right)^{\gamma'}\right),$$

where $\gamma' = 0$ if $\mathcal{U}$ is balanced and $\gamma' = 2$ if it is unbalanced.

Definition

A critical update family is *rooted* if it has two non-opposite hard directions and *unrooted* otherwise. Families with one hard direction are *semi-directed*, while those with no hard directions are *isotropic*.

Definition

A critical update family is *rooted* if it has two non-opposite hard directions and *unrooted* otherwise. Families with one hard direction are *semi-directed*, while those with no hard directions are *isotropic*.

Theorem (MMT19, H–Marêché'21+, H'21+)

For critical $\mathcal{U}$-KCM with difficulty α we have

$$\tau_0 = \exp\left(\Theta(1)\left(\frac{1}{p^\alpha}\right)^\beta\left(\log\frac{1}{p}\right)^\gamma\left(\log\log\frac{1}{p}\right)^\delta\right),$$

where the exponents β, γ, δ are given in the table below.

Definition

A critical update family is *rooted* if it has two non-opposite hard directions and *unrooted* otherwise. Families with one hard direction are *semi-directed*, while those with no hard directions are *isotropic*.

Theorem (MMT19, H–Marêché'21+, H'21+)

For critical $\mathcal{U}$-KCM with difficulty α we have

$$\tau_0 = \exp\left(\Theta(1)\left(\frac{1}{p^\alpha}\right)^\beta\left(\log\frac{1}{p}\right)^\gamma\left(\log\log\frac{1}{p}\right)^\delta\right),$$

where the exponents β, γ, δ are given in the table below.

	Infinite stable dir.	Finite stable dir.	
		Rooted	Unrooted
Unbalanced	$2,4,0$	$1,3,0$	$1,2,0$
Balanced	$2,0,0$	$1,1,0$	$1,0,1$ S.-dir. / Iso. $1,0,0$

Beyond universality

Beyond universality

Theorem (Gravner–Holroyd'08,H–Morris'19)

For the 2-neighbour bootstrap percolation we have

$$\tau_0 = \exp\left(\frac{\pi^2}{18p} - \frac{\Theta(1)}{\sqrt{p}}\right).$$

Beyond universality

Theorem (Gravner–Holroyd'08,H–Morris'19)

For the 2-neighbour bootstrap percolation we have

$$\tau_0 = \exp\left(\frac{\pi^2}{18p} - \frac{\Theta(1)}{\sqrt{p}}\right).$$

Theorem (H–Martinelli–Toninelli20+)

For the 2-neighbour KCM we have

$$\tau_0 = \exp\left(\frac{\pi^2}{9p} + \frac{O\left(\log^{O(1)}(1/p)\right)}{\sqrt{p}}\right).$$

Higher dimensions

Higher dimensions

Theorem (3 × Balister–Bollobás–Morris–Smith'22)

Bootstrap percolation universality statements for supercritical and subcritical families extend to higher dimensions modulo adapting the definition as needed. For every critical family there exists an integer $1 \leqslant r \leqslant d-1$ such that

$$\tau_0 = \exp^{\circ r}\left(p^{-\Theta(1)}\right).$$

Higher dimensions

Theorem (3 × Balister–Bollobás–Morris–Smith'22)

Bootstrap percolation universality statements for supercritical and subcritical families extend to higher dimensions modulo adapting the definition as needed. For every critical family there exists an integer $1 \leqslant r \leqslant d-1$ *such that*

$$\tau_0 = \exp^{\circ r}\left(p^{-\Theta(1)}\right).$$

For KCM the analogous universality result (with a rooted/unrooted distinction for supercritical families) is not known. More precisely, the upper bounds for supercritical and critical families are still missing.

Open problems

Open problems

- Establish the KCM universality in higher dimensions.

Open problems

- Establish the KCM universality in higher dimensions.
- Determine the higher dimensional analogue of α, when this is possible.

Open problems

- Establish the KCM universality in higher dimensions.
- Determine the higher dimensional analogue of α, when this is possible.
- Prove that critical bootstrap percolation models in two dimensions have a sharp threshold.

Open problems

- Establish the KCM universality in higher dimensions.
- Determine the higher dimensional analogue of α, when this is possible.
- Prove that critical bootstrap percolation models in two dimensions have a sharp threshold.
- Find the asymptotics of τ_0 starting from a product measure different from the invariant one. Even for the 1-neighbour case this is open when p is not close to 1.

Open problems

- Establish the KCM universality in higher dimensions.
- Determine the higher dimensional analogue of α, when this is possible.
- Prove that critical bootstrap percolation models in two dimensions have a sharp threshold.
- Find the asymptotics of τ_0 starting from a product measure different from the invariant one. Even for the 1-neighbour case this is open when p is not close to 1.
- Prove the sharpness of the phase transition of subcritical models, even in 2 dimensions.

Open problems

- Establish the KCM universality in higher dimensions.
- Determine the higher dimensional analogue of α, when this is possible.
- Prove that critical bootstrap percolation models in two dimensions have a sharp threshold.
- Find the asymptotics of τ_0 starting from a product measure different from the invariant one. Even for the 1-neighbour case this is open when p is not close to 1.
- Prove the sharpness of the phase transition of subcritical models, even in 2 dimensions.
- Determine or even conjecture which subcritical models exhibit a continuous phase transition.

Open problems

- Establish the KCM universality in higher dimensions.
- Determine the higher dimensional analogue of α, when this is possible.
- Prove that critical bootstrap percolation models in two dimensions have a sharp threshold.
- Find the asymptotics of τ_0 starting from a product measure different from the invariant one. Even for the 1-neighbour case this is open when p is not close to 1.
- Prove the sharpness of the phase transition of subcritical models, even in 2 dimensions.
- Determine or even conjecture which subcritical models exhibit a continuous phase transition.
- Prove anything about the behaviour of general subcritical KCM, which is not witnessed in bootstrap percolation.

Bibliography

[1] D. Aldous and P. Diaconis, *The asymmetric one-dimensional constrained Ising model: rigorous results*, J. Stat. Phys. **107** (2002), no. 5-6, 945–975 pp. MR1901508

[2] P. Balister, B. Bollobás, R. Morris, and P. Smith, *The critical length for growing a droplet*, arXiv e-prints (2022), available at arXiv:2203.13808.

[3] P. Balister, B. Bollobás, R. Morris, and P. Smith, *Subcritical monotone cellular automata*, arXiv e-prints (2022), available at arXiv:2203.01917.

[4] P. Balister, B. Bollobás, R. Morris, and P. Smith, *Universality for monotone cellular automata*, arXiv e-prints (2022), available at arXiv:2203.13806.

[5] P. Balister, B. Bollobás, M. Przykucki, and P. Smith, *Subcritical $\mathcal{U}$-bootstrap percolation models have non-trivial phase transitions*, Trans. Amer. Math. Soc. **368** (2016), no. 10, 7385–7411 pp. MR3471095

[6] B. Bollobás, H. Duminil-Copin, R. Morris, and P. Smith, *Universality of two-dimensional critical cellular automata*, Proc. Lond. Math. Soc. (To appear).

[7] B. Bollobás, P. Smith, and A. Uzzell, *Monotone cellular automata in a random environment*, Combin. Probab. Comput. **24** (2015), no. 4, 687–722 pp. MR3350030

[8] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Kinetically constrained spin models*, Probab. Theory Related Fields **140** (2008), no. 3-4, 459–504 pp. MR2365481

[9] F. Chung, P. Diaconis, and R. Graham, *Combinatorics for the East model*, Adv. Appl. Math. **27** (2001), no. 1, 192–206 pp. MR1835679

[10] J. Gravner and D. Griffeath, *Scaling laws for a class of critical cellular automaton growth rules*, Random walks (Budapest, 1998), 1999, 167–186 pp. MR1752894

[11] J. Gravner and A. E. Holroyd, *Slow convergence in bootstrap percolation*, Ann. Appl. Probab. **18** (2008), no. 3, 909–928 pp. MR2418233

[12] I. Hartarsky, *$\mathcal{U}$-bootstrap percolation: critical probability, exponential decay and applications*, Ann. Inst. Henri Poincaré Probab. Stat. **57** (2021), no. 3, 1255–1280 pp. MR4291442

[13] I. Hartarsky, *Refined universality for critical KCM: upper bounds*, arXiv e-prints (2021), available at arXiv:2104.02329.

[14] I. Hartarsky, *Bootstrap percolation, probabilistic cellular automata and sharpness*, J. Stat. Phys. (To appear).

[15] I. Hartarsky and L. Marêché, *Refined universality for critical KCM: lower bounds*, Combin. Probab. Comput. (To appear).

[16] I. Hartarsky, L. Marêché, and C. Toninelli, *Universality for critical KCM: infinite number of stable directions*, Probab. Theory Related Fields **178** (2020), no. 1, 289–326 pp. MR4146539

[17] I. Hartarsky, F. Martinelli, and C. Toninelli, *Universality for critical KCM: finite number of stable directions*, Ann. Probab. **49** (2021), no. 5, 2141–2174 pp. MR4317702

[18] I. Hartarsky, F. Martinelli, and C. Toninelli, *Sharp threshold for the FA-2f kinetically constrained model*, Probab. Theory Related Fields (To appear).

[19] I. Hartarsky and T. R. Mezei, *Complexity of two-dimensional bootstrap percolation difficulty: algorithm and NP-hardness*, SIAM J. Discrete Math. **34** (2020), no. 2, 1444–1459 pp. MR4117299

[20] I. Hartarsky and R. Morris, *The second term for two-neighbour bootstrap percolation in two dimensions*, Trans. Amer. Math. Soc. **372** (2019), no. 9, 6465–6505 pp. MR4024528

[21] I. Hartarsky and R. Szabó, *Subcritical bootstrap percolation via Toom contours*, arXiv e-prints (2022), available at arXiv:2203.16366.

[22] L. Marêché, *Combinatorics for general kinetically constrained spin models*, SIAM J. Discrete Math. **34** (2020), no. 1, 370–384 pp. MR4062795

[23] L. Marêché, F. Martinelli, and C. Toninelli, *Exact asymptotics for Duarte and supercritical rooted kinetically constrained models*, Ann. Probab. **48** (2020), no. 1, 317–342 pp. MR4079438

[24] F. Martinelli, R. Morris, and C. Toninelli, *Universality results for kinetically constrained spin models in two dimensions*, Comm. Math. Phys. **369** (2019), no. 2, 761–809 pp. MR3962008

[25] F. Martinelli and C. Toninelli, *Towards a universality picture for the relaxation to equilibrium of kinetically constrained models*, Ann. Probab. **47** (2019), no. 1, 324–361 pp. MR3909971

[26] F. Mauch and J. Jäckle, *Recursive dynamics in an asymmetrically constrained kinetic Ising chain*, Phys. A **262** (1999), no. 1-2, 98–117 pp.

[27] A. Shapira, *A note on the spectral gap of the Fredrickson–Andersen one spin facilitated model*, J. Stat. Phys. **181** (2020), no. 6, 2346–2352 pp. MR4179809

[28] P. Sollich and M. R. Evans, *Glassy time-scale divergence and anomalous coarsening in a kinetically constrained spin chain*, Phys. Rev. Lett. **83** (1999), no. 16, 3238–3241 pp.

Thank you.

?