Percolation on the cubic lattice with lower-dimensional disorder

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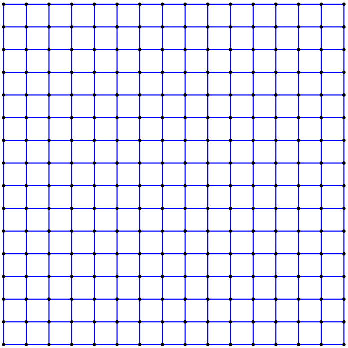
Colmea and SABP – 16 February 2022

Percolation

- $\omega(e) = 1 \quad \rightarrow \quad e$ open (present).
- $\omega(e) = 0 \quad \rightarrow \quad e$ closed (removed).



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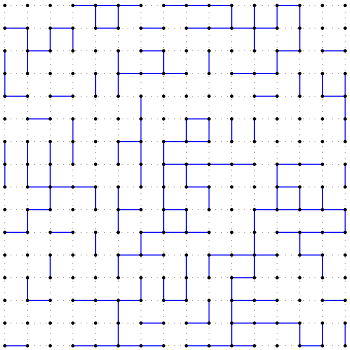
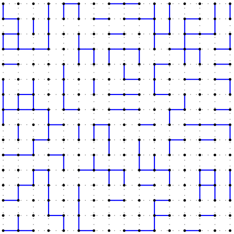


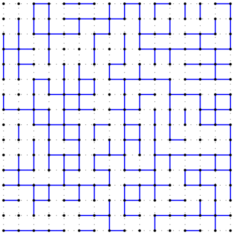
Bernoulli percolation

(Broadbent and Hammerseley '57)

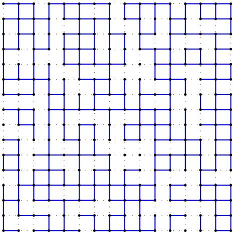
Open with prob. p (closed with prob. $1-p$), independently.



$p = 0.35$

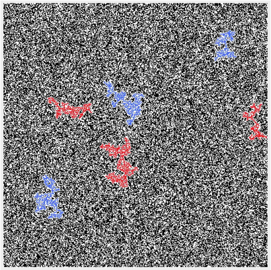
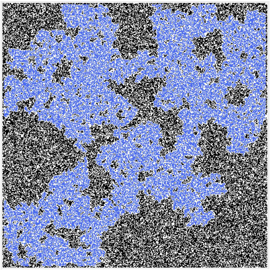
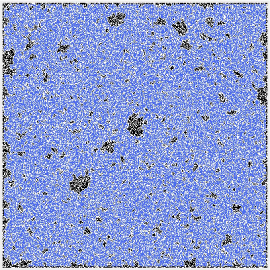


$p = 0.50$



$p = 0.65$

The sharpness of the phase transition

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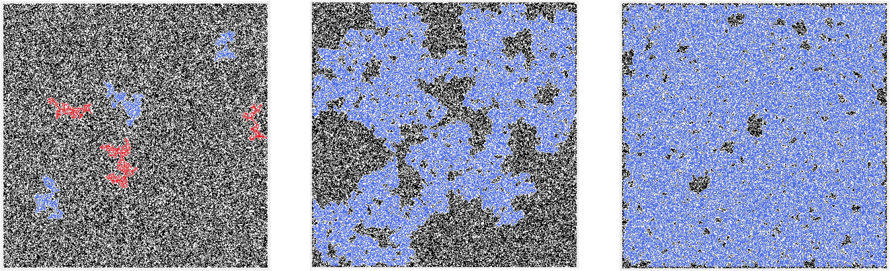

$$\theta(p) = \mathbb{P}_p(o \leftrightarrow \infty), \quad p_c = \sup\{p \colon \theta(p) = 0\}, \quad \pi(p,n) = \mathbb{P}_p(o \leftrightarrow \partial B(n)).$$

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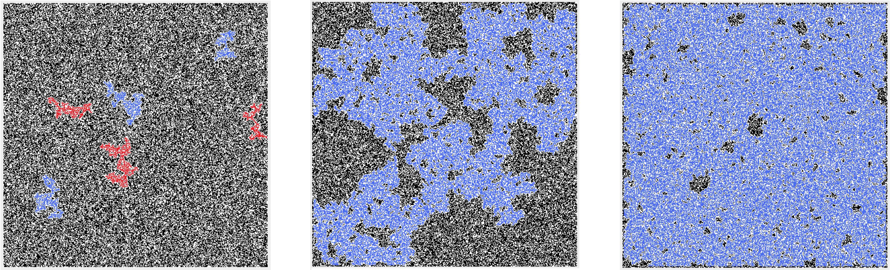

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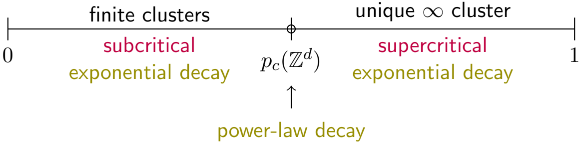


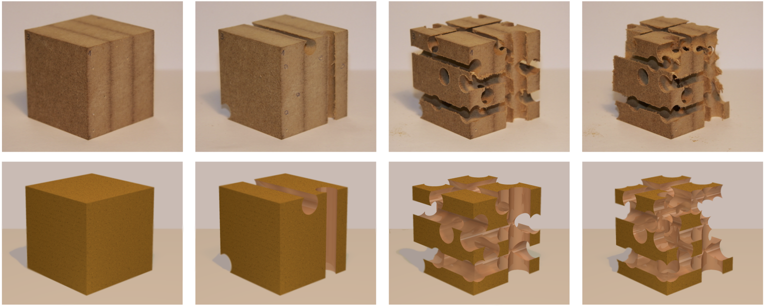
Percolation with lower-dimensional disorder

What happens in the presence of lower-dimensional disorder?
e.g. when:

- edges along the same column are strongly correlated;
- p changes from column to column (randomly);
- the lattice is dilute by removing some of the columns.

Techniques: control of environment and percolation with renormalization.

Drilling through a wooden cube



Drilled by Nuno Araújo and colleages. Figure taken from Phys. Rev. Lett. 116 55701 (2016)

Y. Kantor, Three-dimensional percolation with removed lines of sites, Phys. Rev. B, (1986).

K.J. Schrenk et. al., Phys. Rev. Lett. 116 55701 (2016).

Bernoulli line percolation

Drilling straight lines in the Euclidean lattice

- $\mathbb{Z}^d$, $d \geq 3$, $\{e_1, \dots, e_d\}$.
- $\mathbf{p} = (p_1, \dots, p_d) \qquad p_i \in (0, 1)$.
- **Remove** lines parallel to e_i with probability $1 - p_i$ indep.
- $\mathbb{P}_{\mathbf{p}} =$ probab. dist.

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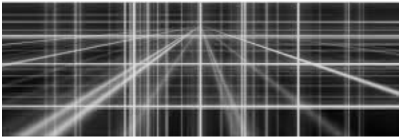


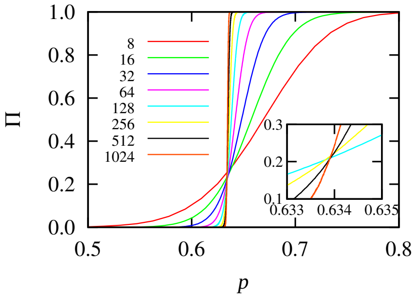
Goal: study connectivity properties of the remaining lattice

Phase transition

Theorem (H., Sidoravicius (2019))

- *If* $p_1 < p_c(\mathbb{Z}^{d-1})$*, then* $\mathbb{P}_{\mathbf{p}}(o \leftrightarrow \infty) = 0$*.*
- *If* **all** $p_1, \ldots, p_d$ *are sufficiently* **large***, then* $\mathbb{P}_{\mathbf{p}}(o \leftrightarrow \infty) > 0$*.*

$d = 3$, $p_1 = p_2 = p_3 = p$. Critical point: $p_* = 0.6339(5)$



(Simulations in PRL 116, 55701)

Conncetivity decay and sharpness

Theorem (H., Sidoravicius (2019))

- *If p_1 and p_2 are sufficiently* **small**, *then*

$$\mathbb{P}_{\mathbf{p}}\big(o \leftrightarrow \partial B(n)\big) \leq e^{-\psi n}.$$

- *If $p_2, \ldots, p_d$ are* **large** *enough,*

$$\mathbb{P}_{\mathbf{p}}\big(o \leftrightarrow \partial B(n),\ o \not\leftrightarrow \infty\big) \geq \alpha_1 n^{-\alpha_2}.$$

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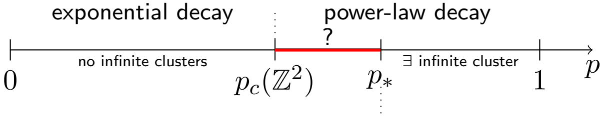
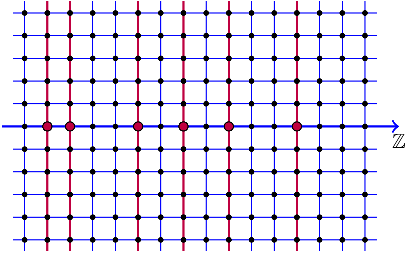
$d = 3$:
$\max\{p_1, p_2\} < p_c(\mathbb{Z}^2)$: **exponential**.
$\min\{p_1, p_2\} > p_c(\mathbb{Z}^2)$: **power-law**.



Columnar inhomogeneities

- Select vertical columns: (B: brochettes, defects, enhancements,...)



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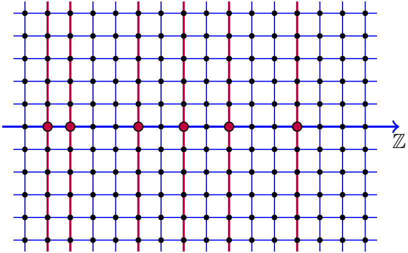

- open e with prob. $= \begin{cases} q, & \text{in a selected column}, \\ p, & \text{otherwise.} \end{cases}$

Some special well-know cases

$$P_{q,p} = P^{B}_{q,p}$$

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$$P_{p_c+\varepsilon,p_c-\delta}(o \leftrightarrow \infty) > 0 \quad \text{iff} \quad \varepsilon > \delta.$$

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- Aizenman & Grimmett, '91 – B: **regularly** spaced:

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \;\; \text{s.t. } P_{p_c+\varepsilon,p_c-\delta}(o \leftrightarrow \infty) > 0.$$

What if B is random?

Assume B are Geom(ρ) spaced

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$$\forall\, \varepsilon \text{ and } \rho > 0,\ \exists\, \delta > 0 \ \text{ s.t. } P_{p_c+\varepsilon, p_c-\delta}(o \leftrightarrow \infty) > 0.$$

In words:
Even when good is just slightly good and very rare, if bad is not too bad, percolation still occurs.

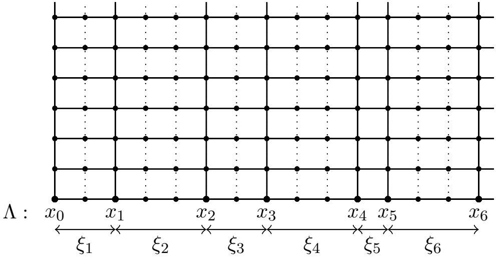
Three different kinds of result

B are Geom(ρ) spaced

- Bramson, Durrett, Schonmann (1991):
  bad lines: **very bad**
  good lines: **rare**
  if good is **sufficiently good**: survival.
- Kesten, Sidoravicius, Vares (2012):
  bad lines: **very bad**
  good lines: **just slightly good**
  if bad lines are **sufficiently rare**: cannot disrupt percolation.
- Duminil-Copin, H., Kozma, Sidoravicius (2018):
  good lines: just **slightly good** and **very rare**
  if bad lines are **not too bad**: cannot disrupt percolation.

Dilute lattices

- ξ integer-valued. $\xi_1, \xi_2, \ldots$ independent copies.
- remove vertical edges outside $\Lambda \times \mathbb{Z}$.



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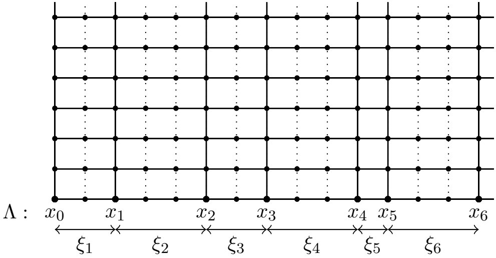
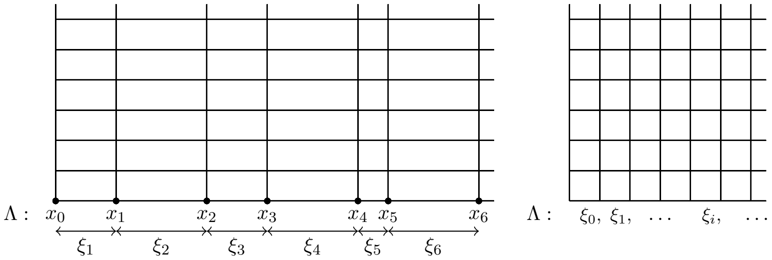


- remaining edges: open indep. with prob. p.

Viewed as lower-dimensional disorder



- Conditioned on ξ_i's:

$$p_e = \begin{cases} p, & e \text{ vertical} \\ p^{\xi_{i+1}}, & e \text{ horizontal} \end{cases}$$

- Phase transition? How does it depend on ξ?

Jonasson, Mosel, Peres (2000), Hoffman (2005): $\xi \sim$ geom.

Phase transition and moments of ξ

Theorem (H., Sá, Sanchis, Teixeira 20+)

- $\mathbb{E}(\xi^{1+\varepsilon}) < \infty$ *for some* $\varepsilon > 0$:

  There exists $p_c \in (0,1)$ *s.t.*

$$p > p_c \implies P_p^\Lambda(o \leftrightarrow \infty) > 0, \; for \; a.e. \; \Lambda.$$

$$p < p_c \implies P_p^\Lambda(o \leftrightarrow \infty) = 0, \; for \; a.e. \; \Lambda.$$

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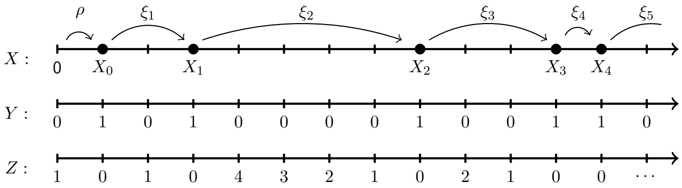
- $\mathbb{E}(\xi^{1-\varepsilon}) = \infty$ *for some* $\varepsilon > 0$*:*

  For any $p \in [0,1)$*,*

$$P_p^\Lambda(o \leftrightarrow \infty) = 0, \; for \; almost \; all \; \Lambda.$$

Proof: separate control of environment and percolation.

Environment as a renewal processes – Decoupling



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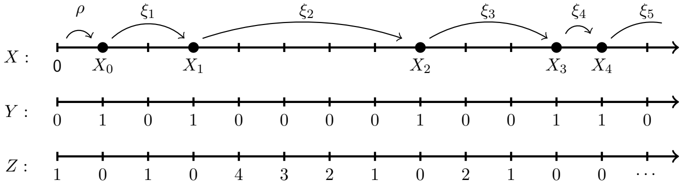
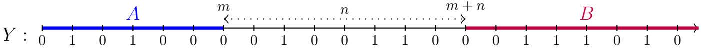


- $A \in \sigma(Y_i;\ 0 \le i \le m)$ $B \in \sigma(Y_i,\ i \ge m+n)$

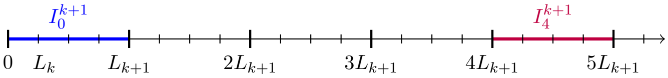


$$P(A \cap B) \le P(A)P(B) + cn^{-\varepsilon}.$$

Control of the environment
Scales and blocks

Scales: $L_0, L_1, L_2, \ldots$

- L_0 large, $L_{k+1} = L_k^{\gamma}$, $\qquad \gamma \in (1,2)$.

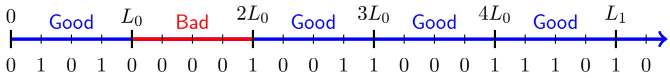


Control of the environment

Bad blocks

Scale 0:

Bad: If no zeroes inside (does not intersect Λ).



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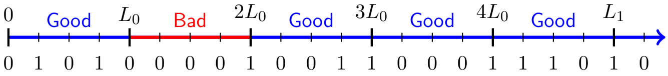
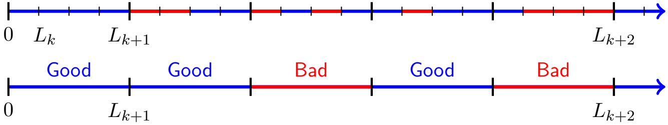


Scale $k+1$:

Bad: two **non-consecutive** bad sub-blocks.



Control of the environment

Rareness of bad blocks

$$p_k := P(\text{a block at scale } k \text{ is bad})$$

Recursion Inequality:

$$p_{k+1} \leq \left(\frac{L_{k+1}}{L_k}\right)^2 \left[p_k^2 + c_1 L_k^{-\varepsilon}\right].$$

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Trigger: L_0 large.

Conclusion: $p_k \leq L_k^{-\alpha}$ for every k.

Control of the percolation

Crossing events

Height scales: $H_0, H_1, \ldots$

$$H_k = 2\lceil \exp(L_k^{\mu}) \rceil H_{k-1}, \quad \mu \in \left(\frac{1}{\gamma}, 1\right)$$

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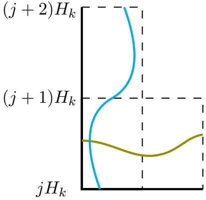

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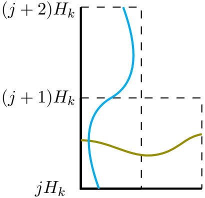
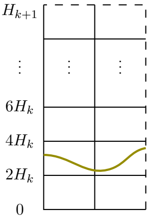


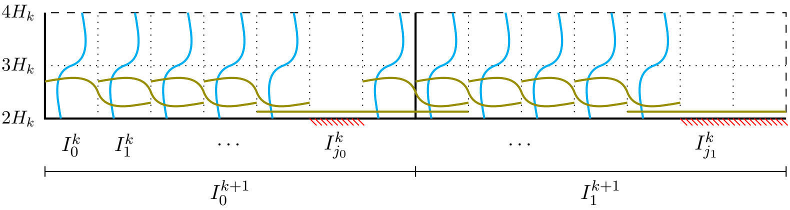
$$q_k(p) := \max_{\Lambda} \left\{ P\big(\text{no crossings when environment good}\big) \right\}.$$

Control of the percolation

Horizontal crossings

$$q_k(p) \leq \exp(-L_k^\beta) \;\Rightarrow\; P\big((C_{0,0}^{k+1})^c\big) \leq \exp(-L_{k+1}^\beta).$$





Control of the percolation

Vertical crossings

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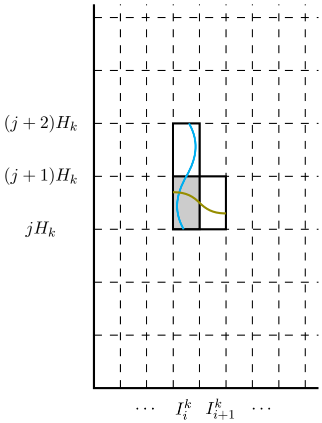

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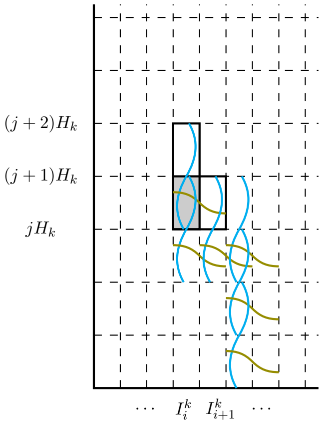
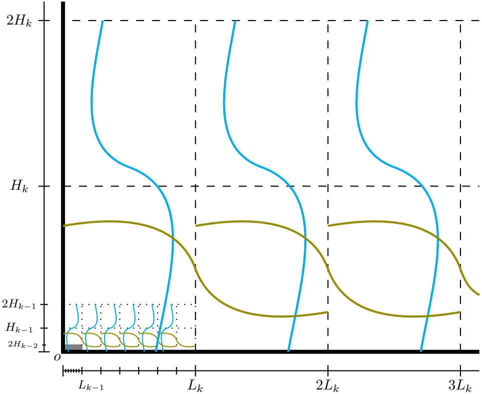


Percolation for p large



- If k large: only good $(k-1)$-blocks in I_0^k.
- Crossings very likely.
- Build the infinite cluster from the crossings.