# Complex Gaussian Multiplicative Chaos and Gaussian Multiplicative Cascade

joint with R. Rhodes (Aix-Marseille University) and V. Vargas (ENS)

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# Plan of the talk

(A) Present a motivation for Gaussian Multiplicative Chaos:
*Liouville 2D-Quantum Gravity* (on slides),

(B) Present a simpler model with the same behavior as GMChaos:
*Gaussian Multiplicative Cascades*,

(C) Look at these model in the complex setup:
*The Sine-Gordon Model*.

# 2D-LQG: What is it? Why is it called gravity?

**2D Liouville Quantum Gravity is two dimensional random geometry**

Its purpose is to construct a physically relevant notion of random 2D (Riemannian) manifold.

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# Choosing a 2D manifold at random: in a discrete setup

Consider $n$ equilateral triangles (equiped with their Euclidean metric). Match pairs ot triangle sides and glue then together. Choose the matching uniformly among the possibilites that produce a manifold with no holes.

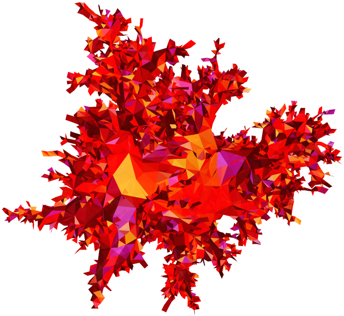

Figure: A large uniform random triangulation

# Choosing a random surface with the topology of the sphere in a continuous fashion

Let us start with the Euclidean sphere $\mathbb{S}_2$ (and let $g_0$ denote its metric tensor). We are going to choose a random metric on $\mathbb{S}_2$ with the conformal class of $g_0$

$$[g_0] := \{g \ : \ g = e^{\varphi} g_0 \ , \ \varphi \text{ scalar field}\}.$$

We want to choose the conformal the conformal factor $\varphi$ at with distribution

$$e^{-H(\varphi)} D\varphi,$$

where $H(\cdot)$ is an "energy functional", and $D\varphi$ is "Lebesgue measure" on the set of scalar functions on $\mathbb{S}_2$.

**Natural choice for $H(\varphi)$**

Chose $H$ from natural minimization problems appearing in "classical" geometry.

# Liouville Quantum Gravity

Given $\mu, \gamma > 0$ we consider the Classical Liouville action which is associated with the curvature homogenization problem.
It is defined by

$$\mathcal{S}_L(\omega) := \frac{1}{4\pi} \int_{\mathbb{S}_2} (|\nabla \omega|^2 + \frac{4}{\gamma}\omega - 4\pi\mu e^{\gamma\omega})\,\mathrm{d}x,$$

If $\omega$ is a local extremum of $\mathcal{S}_L(\omega)$ then the metric associated with $e^{\gamma\omega} g_0$ has uniform curvature $2\pi\mu\gamma^2$.

**Liouville Quantum Gravity [Polyakov '81]**

corresponds to the metric $e^{\gamma\omega} g_0$ where $\gamma \in (0,2)$ is a fixed parameter and $\omega$ is random and with distribution

$$e^{-\frac{1}{4\pi}\int_{\mathbb{S}_2}(|\nabla\omega|^2 + 2\left(\frac{2}{\gamma}+\frac{\gamma}{2}\right)\omega - 4\pi\mu e^{\gamma\omega})\,\mathrm{d}x} D\omega$$

# How to make sense of this?

We have to find first an interpretation for $e^{-\frac{1}{4\pi}\int_{\mathbb{S}_2}|\nabla\omega|^2\,\mathrm{d}x}D\omega$, as a measure.

If $\omega := \sum_{i=1}^{\infty}\alpha_i f_i$ for an orthogonal base $(f_i)_{i\geq 0}$ of eigenfunction of $\Delta$ on $\mathbb{L}^2(\mathbb{S}_2)$ with eigenvalue $-\lambda_i$, ($\lambda_0 = 0$, $\lambda_i > 0$ for $i \geq 1$), we have

$$\int_{\mathbb{S}_2}|\nabla\omega|^2\,\mathrm{d}x = \sum_{i\geq 1}\lambda_i\alpha_i^2$$

Interpreting $D\omega$ as a "product Lebesgue measure on $\mathbb{L}^2(\mathbb{S}_2)$"

$$e^{-\frac{1}{4\pi}\int_{\mathbb{S}_2}|\nabla\omega|^2\,\mathrm{d}x}D\omega = \mathrm{d}\alpha_0 \times \left(\prod_{i=1}^{\infty} e^{-\frac{1}{2\pi}\sum_{i=1}^{\infty}\frac{\lambda_i\alpha_i^2}{2}}\,\mathrm{d}\alpha_i\right). \tag{1}$$

The Fourier coefficients of $\omega$ should be independent Gaussian with variance $(2\pi\lambda_i^{-1})_{i\geq 1}$.

# How to make sense of this? The Gaussian Free Field

The Gaussian Free Field is a random field on $\mathbb{S}_2$ formally defined by

$$X(x) := \sqrt{2\pi} \sum_{i=1}^{\infty} \lambda_i^{-1/2} Z_i \ f_i(x) \quad (*)$$

Where $(Z_i)_{i\geq 0}$ are IID standard Gaussians. The covariance function associated with this formal sum is given by

$$G(x,y) := 2\pi \sum_{i\geq 0} \lambda_i f_i(x) f_i(y).$$

We have $G(x,y) = L(x,y) - \log d(x,y)$. where $L$ is a continuous bounded function. In particular $G(x,x) = \infty$.
While $(*)$ does not converge, $X$ can make sense integrated against test function, if $g \in \mathbb{L}_2(\mathbb{S}_2)$, the sum

$$\int_{\mathbb{S}_2} X g \, \mathrm{d}x = \langle X, g \rangle := \sqrt{2\pi} \sum_{i=1}^{\infty} \lambda_i^{-1/2} Z_i \langle f_i, g \rangle_{\mathbb{L}_2(\mathbb{S}_2)}$$

converges almost surely.

# Interpreting $e^{\gamma X}$/Gaussian Multiplicative Chaos

Once we know that $e^{-\frac{1}{4\pi}\int_{\mathbb{S}_2}|\nabla\omega|^2\,\mathrm{d}x}D\omega$, is to be interpreted the GFF, we need to interpret $e^{\gamma X}\,\mathrm{d}x$ (appearing in $\mathcal{S}_L(\omega)$ and in the metric tensor).

We can look at an approximation for $X$ and consider

$$X_n(x) := \sqrt{2\pi}\sum_{i=1}^{n}\lambda_i^{-1/2}Z_i\ f_i(x)$$

and consider $M_n(\mathrm{d}x) := e^{\gamma X_n(x)-\frac{\gamma^2}{2}\mathbb{E}[X_n^2]}\,\mathrm{d}x$.

**Theorem (Kahane'85, Robert,Vargas'10, Shamov'16, Berestycki'17)**

*If $\gamma\in(0,2)$, there exists a non-trivial random measure $M(\mathrm{d}x)$ such that for every Borel set $A$ we have almost surely*

$$\lim_{n\to\infty}M_n(A) = M(A),$$

Starting with [Duplantier/Sheffield '10] this construction has been the base of mathematically rigourous 2D-LQG.

# A moral justification of such a construction

It is conjectured that when $n \to \infty$, after appropriate renormalization, the distribution of large uniform triangulation converge to 2D-LQG corresponding to $\gamma = \sqrt{8/3}$.

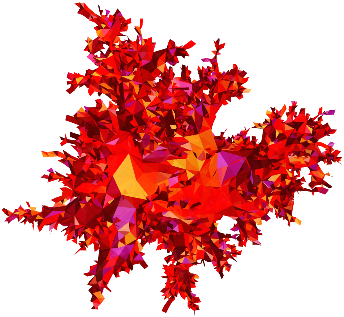

Figure: A large random triangulation (same as before)