An approach to Metastability

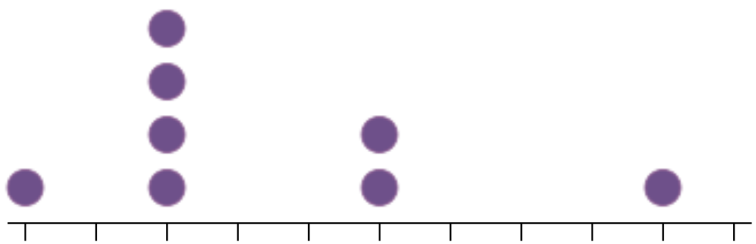
J. Beltran, A. Gaudillière, B. Gois, M. Jara, R. Misturini, A. Teixeira

Model 1: zero range processes

- $\mathbb{T}_L = \{1, \ldots, L\}$
- State space $\mathbb{N}^{\mathbb{T}_L}$
- configurations $\eta = \{\eta_x : x \in \mathbb{T}_L\}$

Dynamics

- $g : \mathbb{N} \to \mathbb{R}_+ \quad g(0) = 0 \qquad g(k) > 0 \qquad 0 < p \leq 1$
- $x \to x + 1$ at rate $pg(\eta_x) \qquad x \to x - 1$ at rate $(1 - p)g(\eta_x)$
- $g(k) = k$ independent random walks
- $g(k) = \mathbf{1}\{k \geq 1\}$ queues and servers
- $g \downarrow$ sticky



Model 2: Random walks among random traps

- $\{G_N : N \geq 1\}$ $\quad G_N = (V_N, E_N)$ $\quad$ finite graphs
- $G_N = \mathbb{T}_N^d$ $\quad \{0,1\}^N$
- Random d-regular graphs $\quad$ Super-critical Erdös-Rényi

Model 2: Random walks among random traps

- $\{G_N : N \geq 1\}$ $\quad G_N = (V_N, E_N)$ $\quad$ finite graphs
- $G_N = \mathbb{T}_N^d$ $\quad \{0,1\}^N$
- Random d-regular graphs $\quad$ Super-critical Erdös-Rényi
- $\{W_k^N : k \geq 1\}$ $\quad$ i.i.d.
- Basin of attraction α-stable $\quad 0 < \alpha < 1$
- $\mathbb{P}[W_1^N > t] \;=\; \dfrac{L(t)}{t^\alpha} \quad t > 0$
- $L(t)$ slowly varying at infinity

Model 2

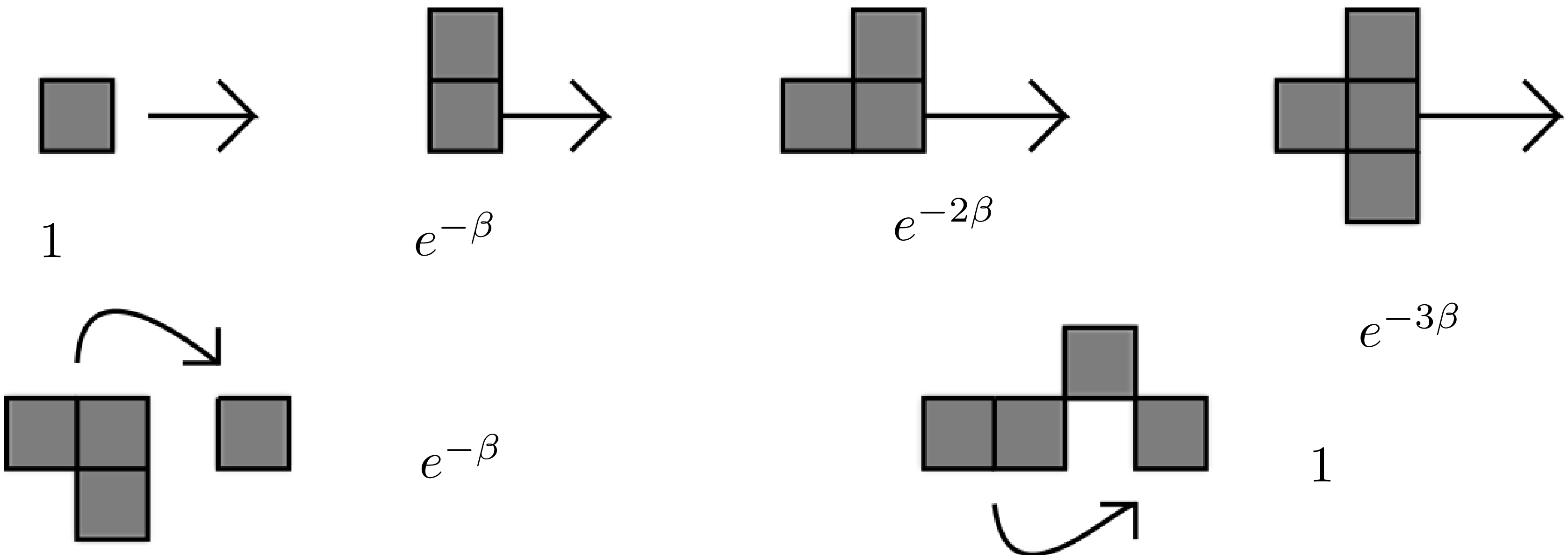
- $\{G_N : N \geq 1\}$ $\quad G_N = (V_N, E_N)$ $\quad$ finite graphs
- $W_j > 0 \quad \sum_{j\geq 1} W_j < \infty$
- $x_1^N, x_2^N, \dots, x_{|V_N|}^N$ random permutation of V_N
- $W^N_{x_j^N} = W_j$
- X_t^N random walk on G_N $\quad$ mean W_x^N exp. times

Model 2

- $\{G_N : N \geq 1\}$ $\quad G_N = (V_N, E_N)$ $\quad$ finite graphs
- $W_j > 0 \quad \sum_{j \geq 1} W_j < \infty$
- $x_1^N, x_2^N, \ldots, x_{|V_N|}^N$ random permutation of V_N
- $W_{x_j^N}^N = W_j$
- X_t^N random walk on G_N $\quad$ mean W_x^N exp. times
- $(\mathcal{L}_N f)(x) \; = \; \frac{1}{\deg(x)} \frac{1}{W_x^N} \sum_{y \sim x} [f(y) - f(x)]$
- $\deg(x)$ degree of x

Model 3: Kawasaki dynamics

- $\mathbb{T}_L = \{-L, \ldots, L\}^2$
- $\Omega_L = \{0, 1\}^{\mathbb{T}_L}$
- $\eta = \{\eta(x) : x \in \mathbb{T}_L\}, \quad \eta(x) = 1, \quad \eta(x) = 0$
- Inverse temperature $\beta \uparrow \infty$



Model 1: Stationary states

- N number of particles
- $E_{L,N} = \{\eta \in \mathbb{N}^{\mathbb{T}_L} : \sum_{x \in \mathbb{T}_L} \eta_x = N\}$
- $\{\eta(t) : t \geq 0\}$ irreducible
- Exists unique stationary state $\mu_{L,N}$

Model 1: Stationary states

- N number of particles
- $E_{L,N} = \{\eta \in \mathbb{N}^{\mathbb{T}_L} : \sum_{x \in \mathbb{T}_L} \eta_x = N\}$
- $\{\eta(t) : t \geq 0\}$ irreducible
- Exists unique stationary state $\mu_{L,N}$

Equivalence of ensembles:

- Cylinder function f $\quad f = f(\eta_{-m}, \ldots, \eta_m)$
- $\lim_{\substack{L \to \infty \\ N/L \to \rho}} E_{\mu_{L,N}}[f] = E_{\nu_\rho}[f]$
- $\mathbb{N}^{\mathbb{Z}}$ $\quad$ stationary state (Grand canonical)
- Number of particles conserved, $\quad \{\nu_\rho : \rho \geq 0\} \quad E_{\nu_\rho}[\eta_0] = \rho$

Grand canonical stationary states

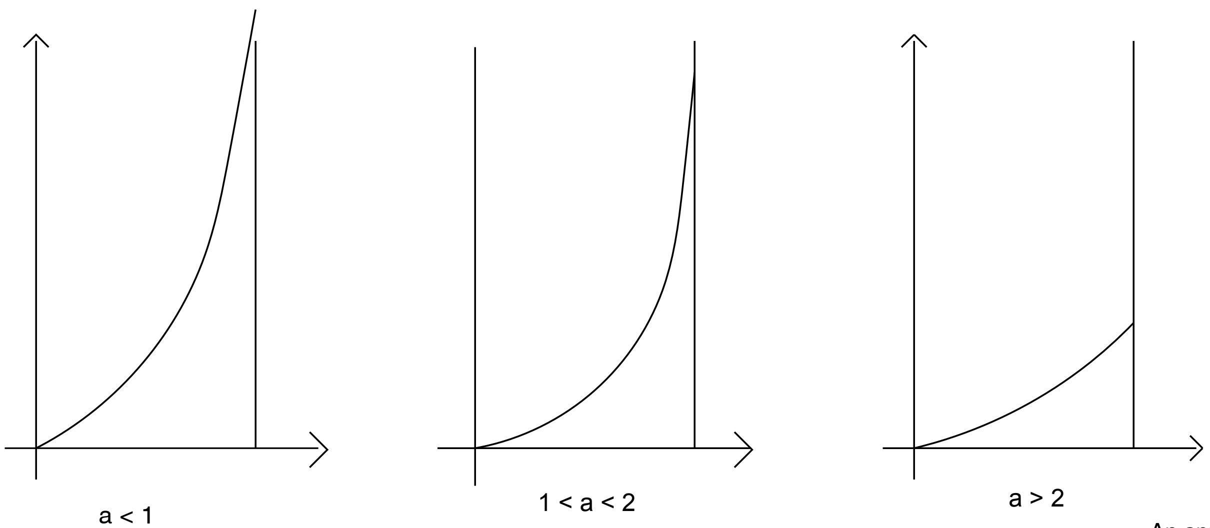
- Partition function: $Z(\varphi) \ = \ \sum_{k \geq 0} \frac{\varphi^k}{g(k)!}\ , \quad \varphi \geq 0$
- $g(0)! \ = \ 1\ , \quad g(k)! \ = \ g(1) \cdots g(k)$
- $g(1) = 1 \quad g(k) = \Big(\frac{k}{k-1}\Big)^{\alpha} \quad k \geq 2 \quad \alpha > 0 \quad g(k)! = k^{\alpha}$
- $\varphi^* < \infty$ radius of convergence of Z $\quad \varphi^* = 1$
- $\varphi < \varphi^* \quad \hat{\nu}_{\varphi}$ product measure on $\mathbb{N}^{\mathbb{Z}}$
- $\hat{\nu}_{\varphi}\{\eta : \eta_x = k\} \ = \ \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!}$

Equivalence of ensembles

- $\hat{\nu}_\varphi\{\eta : \eta_x = k\} \;=\; \dfrac{1}{Z(\varphi)} \dfrac{\varphi^k}{g(k)!} \qquad 0 \le \varphi < \varphi^*$
- $R(\varphi) \;=\; E_{\hat{\nu}_\varphi}[\eta_0] \;=\; \dfrac{1}{Z(\varphi)} \sum_{k\ge 1} k \dfrac{\varphi^k}{g(k)!} \;=\; \dfrac{\varphi Z'(\varphi)}{Z(\varphi)} = \varphi \dfrac{d}{d\varphi} \log Z(\varphi)$
- $R(0) = 0 \qquad R$ strictly increasing
- $\rho^* = \lim_{\varphi\to\varphi^*} R(\varphi) \qquad R : [0, \varphi^*) \to [0, \rho^*) \qquad \Phi = R^{-1}$
- $0 \le \rho < \rho^* \qquad \nu_\rho = \hat{\nu}_{\Phi(\rho)}$
- $E_{\nu_\rho}[\eta_0] \;=\; E_{\hat{\nu}_{\Phi(\rho)}}[\eta_0] \;=\; R(\Phi(\rho)) \;=\; \rho$
- Cylinder function $f \qquad \rho < \rho^* \qquad \lim_{\substack{L\to\infty \\ N/L\to\rho}} E_{\mu_{L,N}}\big[f\big] \;=\; E_{\nu_\rho}\big[f\big]$
- Local central limit theorem (Kipnis - L)

Critical density

- $\rho^* = \lim_{\varphi \to \varphi^*} R(\varphi) = \lim_{\varphi \to \varphi^*} \varphi \frac{d}{d\varphi} \log Z(\varphi) = \varphi^* \lim_{\varphi \to \varphi^*} \frac{d}{d\varphi} \log Z(\varphi)$
- $g(1) = 1 \quad g(k) = \left(\frac{k}{k-1}\right)^\alpha \quad k \geq 2 \quad \alpha > 0 \quad g(k)! = k^\alpha$ sticky
- $Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g(k)!} \quad R(\varphi) = \frac{1}{Z(\varphi)} \sum_{k \geq 0} k \frac{\varphi^k}{g(k)!} \quad \varphi^* = 1$
- $\alpha \leq 1 \quad 1 < \alpha \leq 2 \quad \alpha > 2$



Phase transition

- $\alpha \leq 1 \quad Z(\varphi^*) = \infty \quad \rho^* = \infty$
- $1 < \alpha \leq 2 \quad Z(\varphi^*) < \infty \quad \rho^* = \infty$
- $\alpha > 2 \quad Z(\varphi^*) < \infty \quad \rho^* < \infty$
- Problem: $\mu_{L,N}$ if $N/L = \rho > \rho^*$?

Phase transition

- $\alpha \le 1 \quad Z(\varphi^*) = \infty \quad \rho^* = \infty$
- $1 < \alpha \le 2 \quad Z(\varphi^*) < \infty \quad \rho^* = \infty$
- $\alpha > 2 \quad Z(\varphi^*) < \infty \quad \rho^* < \infty$
- Problem: $\mu_{L,N}$ if $N/L = \rho > \rho^*$?
- $\{N_L : L \ge 1\} \quad N_L/L \to \rho > \rho^* \quad \mu_{L,N} T^{-1} \sim \nu_{\rho^*}$
- $\alpha > 1 \quad L$ fixed
- $1 \ll \ell_N \ll N \quad \lim_{N\to\infty} \mu_{L,N}\{\max_{1\le x\le L} \eta_x \ge N - \ell_N\} = 1$
- Let $N \uparrow \infty$, $\mu_{L,N} T^{-1} \to \nu_{\rho^*}$

Model 2 and 3: Stationary states

- Model 2: $\nu_N(x) \ \sim \ \deg(x) W^N(x)$

Model 2 and 3: Stationary states

- Model 2: $\nu_N(x) \ \sim \ \deg(x) W^N(x)$
- Model 3:
- Irreducible sets: $\Omega_{L,K} = \{\eta \in \Omega_L : \sum_{x \in \mathbb{T}_L} \eta(x) = K\}$
- $- \ \mathbb{H}(\eta) \ = \ \sum_{x \sim y} \eta(x)\eta(y)$
- $\mu_K(\eta) \ = \ \frac{1}{Z_{\beta,K}} e^{-\beta \mathbb{H}(\eta)} \ , \quad \eta \in \Omega_{L,K}$

Ground states

- $\lim_{N\to\infty} \mu_{L,N}\{\max_{1\le x\le L}\ \eta_x \ge N - \ell_N\} = 1$

Ground states

- $\lim_{N\to\infty} \mu_{L,N}\{\max_{1\le x\le L}\ \eta_x \ge N - \ell_N\} = 1$
- $\nu_N(x)\ \sim\ \deg(x) W^N(x)$
- $\nu_N(x_1^N, \ldots, x_M^N) \to 1$

Ground states

- $\lim_{N\to\infty} \mu_{L,N}\{\max_{1\le x\le L}\ \eta_x \ge N - \ell_N\} = 1$
- $\nu_N(x) \ \sim\ \deg(x) W^N(x)$
- $\nu_N(x_1^N, \dots, x_M^N) \to 1$
- $K = n^2 \quad L > 2n \quad \eta^{\mathbf{x}} \quad \Gamma \ = \ \{\eta^{\mathbf{x}} : \mathbf{x} \in \mathbb{T}_L\}$
- $\mu_\beta(\Gamma) \to 1$

Questions

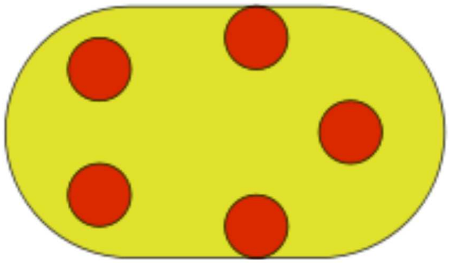
- $\{\eta(t) : t \geq 0\}$
- $\mathcal{E}_1^N, \ldots, \mathcal{E}_{\kappa_N}^N$ meta-sets
- Suppose $\eta(0) \in \mathcal{E}_i^N$

Questions

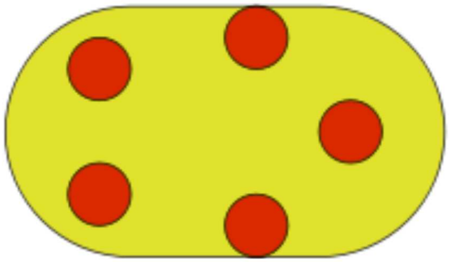
- $\{\eta(t) : t \geq 0\}$
- $\mathcal{E}_1^N, \dots, \mathcal{E}_{\kappa_N}^N$ meta-sets
- Suppose $\eta(0) \in \mathcal{E}_i^N$
- $T_N = \inf\{t > 0 : \eta(t) \in \cup_{j \neq i} \mathcal{E}_j^N\}$
- Order of T_N?
- $P[\eta(T_N) \in \mathcal{E}_j^N]$

Metastability/Tunneling

- $\mathcal{E}_N^x \quad 1 \leq x \leq \kappa_N$
- $\mathcal{E}_N = \bigcup_{x=1}^{\kappa_N} \mathcal{E}_N^x \quad E_{L,N} \; = \; \mathcal{E}_N \; \cup \; \Delta_N$



Metastability

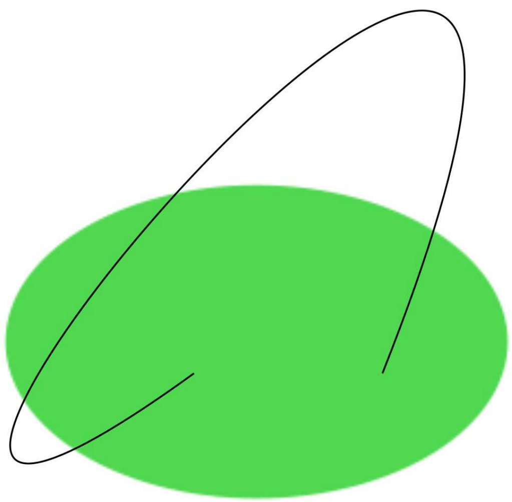


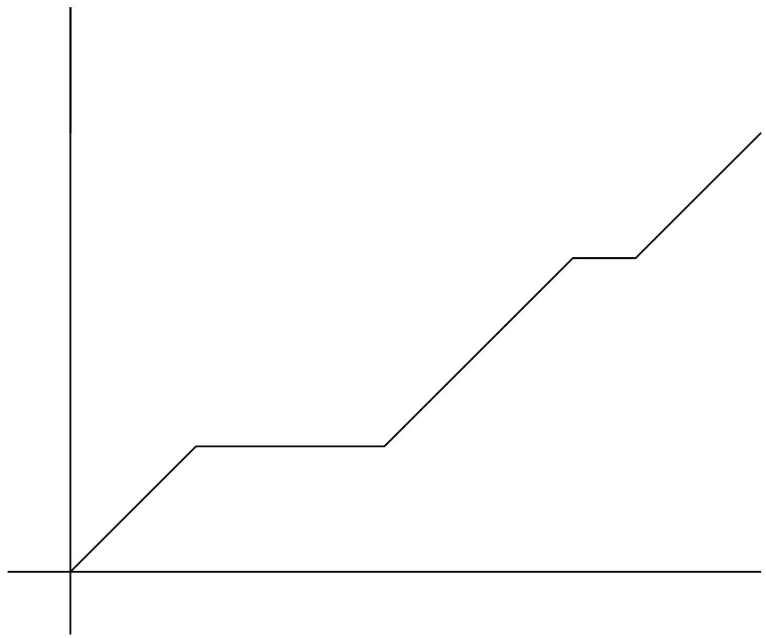
- (M1) Starting from $\mathcal{E}_N^x$, the process thermalizes on $\mathcal{E}_N^x$ before leaving this set.
- (M2) On an appropriate time scale, process jumps from $\mathcal{E}_N^x$ to $\mathcal{E}_N^y$ at exponential times.
- (M3) On that time scale, the time spent on Δ_N is negligible.

A martingale approach to Metastability

Trace

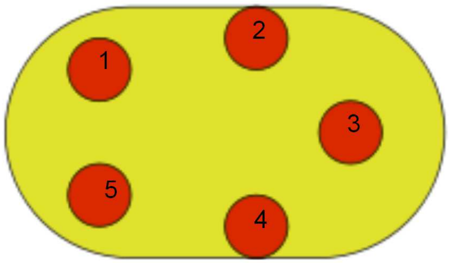
- $\eta^{\mathcal{E}_N}(t)$ trace of $\{\eta(t) : t \geq 0\}$ on $\mathcal{E}_N = \bigcup_{x=1}^{\kappa_N} \mathcal{E}_N^x$:
- $T(t) = \int_0^t \mathbf{1}\{\eta(s) \in \mathcal{E}_N\}\, ds$
- $S(t) = \sup\{s : T(s) \leq t\}$
- $\eta^{\mathcal{E}_N}(t) = \eta(S(t))$ Markov process on $\mathcal{E}_N$





Asymptotic Markovian Dynamics

- $\eta^{\mathcal{E}_N}(t)$ trace of $\{\eta(t) : t \geq 0\}$ on $\mathcal{E}_N = \bigcup_{x=1}^{\kappa_N} \mathcal{E}_N^x$
- $\Psi_N : \mathcal{E}_N \to \{1, \dots, \kappa_N\} \qquad \Psi_N(\eta) = x$ iff $\eta \in \mathcal{E}_N^x$
- $X_N(t) = \Psi_N(\eta^{\mathcal{E}_N}(t))$ may not be Markovian



(M2): $X_N(t\theta_N) \to X(t)$ Markov process on $\{1, \dots, \kappa_N\}$.

Martingale approach

- $X_t^N = \Psi(\eta^{\mathcal{E}}(t\theta_N)) \longrightarrow X_t$
- Tightness X_t^N
- X_t solves martingale problem $\quad F : \{1, \dots, \kappa\} \to \mathbb{R}$
- $F(X_t) - F(X_0) - \int_0^t (\mathcal{L}F)(X_s)\, ds$
- $F(X_t) - F(X_0) - \int_0^t \sum_{y=1}^{\kappa} r(X_s, y)[F(y) - F(X_s)]\, ds$

Martingale approach

- $F(X_t) - F(X_0) - \int_0^t \sum_{y=1}^{\kappa} r(X_s, y)[F(y) - F(X_s)]\, ds$

- $M_t^N = F(X_t^N) - F(\Psi(\eta^{\mathcal{E}}(0))) - \int_0^{t\theta_N} [L_{\mathcal{E}}(F \circ \Psi)](\eta^{\mathcal{E}}(s))\, ds$

$$
\begin{aligned}
[L_{\mathcal{E}}(F \circ \Psi)](\eta) &= \sum_{\xi \in \mathcal{E}_N} R^{\mathcal{E}_N}(\eta, \xi) \{(F \circ \Psi)(\xi) - (F \circ \Psi)(\eta)\} \\
&= \sum_{x,y=1}^{\kappa_N} [F(y) - F(x)] \sum_{\xi \in \mathcal{E}_N^y} R^{\mathcal{E}_N}(\eta, \xi)\, \mathbf{1}\{\eta \in \mathcal{E}_N^x\} \\
&= \sum_{x,y=1}^{\kappa_N} [F(y) - F(x)]\, R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y)\, \mathbf{1}\{\eta \in \mathcal{E}_N^x\}
\end{aligned}
$$

Metastable ergodicity

- $F(X_t^N) - F(X_0^N) - \sum_{x,y=1}^{\kappa_N} [F(y) - F(x)] \int_0^{t\theta_N} R^{\mathcal{E}_N}(\eta(s), \mathcal{E}_N^y) \mathbf{1}\{\eta(s) \in \mathcal{E}_N^x\} \, ds$
- $G_{x,y}(\eta) = R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y) \mathbf{1}\{\eta \in \mathcal{E}_N^x\}$
- $\int_0^{t\theta_N} G_{x,y}(\eta(s)) \, ds$
- $\mathcal{P} = \sigma\{\mathcal{E}_N^x : 1 \le x \le \kappa_N\} \qquad \hat{G}_{x,y} = E_{\mu_N}[G_{x,y}(\eta) | \mathcal{P}]$

$$\int_0^{t\theta_N} \Big\{ G_{x,y}(\eta_s^{\mathcal{E}}) - \hat{G}_{x,y}(\eta_s^{\mathcal{E}}) \Big\} \, ds \longrightarrow 0 \qquad \textbf{(C1)}$$

- $\hat{G}_{x,y}(\eta) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y) =: r_{\mathcal{E}_N}(x, y)$
- $F(X_t^N) - F(X_0^N) - \int_0^t \sum_{y=1}^{\kappa_N} \theta_N \, r_{\mathcal{E}_N}(X_{s\theta_N}^N, y) \, [F(y) - F(X_{s\theta_N}^N)] \, ds$

Asymptotic behavior of rates

- $F(X_t^N) - F(X_0^N) - \int_0^t \sum_{y=1}^{\kappa_N} \theta_N r_{\mathcal{E}_N}(X_{s\theta_N}^N, y)\, [F(y) - F(X_{s\theta_N}^N)]\, ds$

- $r_{\mathcal{E}_N}(x,y) \;=\; \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y)$

$$\theta_N r_{\mathcal{E}_N}(x,y) \;\longrightarrow\; r(x,y) \qquad \textbf{(C2)}$$

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N} \mathbb{E}_\eta^N \Big[\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\}\, ds \Big] \;=\; 0 \qquad \textbf{(C3)}$$

- Nothing is said if $\eta(0) \in \Delta_N$.

Martingale approach

- **Th** (Beltrán, L.): Sufficient conditions for a ergodic Markov process on a countable space to be metastable.
- All conditions are expressed in terms of the measure $\mu_{L,N}$ and capacities.

Potential Theory, Capacity

- Markov process $\{\eta(t) : t \geq 0\}$ on E
- Rates $R(\eta, \xi)$ $\quad \lambda(\eta) = \sum_{\xi \neq \eta} R(\eta, \xi)$ $\quad M(\eta) = \mu(\eta)\,\lambda(\eta)$
- Hitting and return times

$$H_A = \inf\{t > 0 \ ; \ \eta(t) \in A\}$$

$$H_A^+ = \inf\{t > 0 \ ; \ \eta(t) \in A \ \ \exists s < t \ \ \eta(s) \neq \eta(0)\}$$

- Capacity $\quad A$, $B \subset E$, $A \cap B = \varnothing$

$$\mathrm{cap}(A, B) \ = \ \sum_{\eta \in A} M(\eta)\, \mathbb{P}_\eta[H_B^+ < H_A^+]$$

Dirichlet principle

- Generator L ,   Dirichlet form $D(f) = \langle (-L)f, f \rangle_\mu$

Dirichlet principle

- Generator L ,   Dirichlet form $D(f) = \langle (-L)f, f \rangle_\mu$
- Reversible
- $\mathrm{cap}(A, B) \;=\; \inf_F \langle F, (-L)F \rangle_\mu \;=\; D(V_{A,B})$
- $F_A = 1,\, F_B = 0$
- $V_{A,B}(\eta) = \mathbb{P}_\eta[H_A < H_B]$.

Dirichlet principle

- Generator L ,   Dirichlet form $D(f) = \langle (-L)f, f\rangle_\mu$
- Reversible
- $\mathrm{cap}(A,B) \; = \; \inf_F \langle F, (-L)F\rangle_\mu \; = \; D(V_{A,B})$
- $F_A = 1, \, F_B = 0$
- $V_{A,B}(\eta) = \mathbb{P}_\eta[H_A < H_B]$.
- Non-reversible (Pinsky, Doyle, Gaudillière-L.)
- $\mathrm{cap}(A,B) \; = \; \inf_F \sup_H \Big\{ 2\langle F \, , \, LH\rangle_\mu \; - \; \langle H, (-L)H\rangle_\mu \Big\}$,
- $F_A = 1, \, F_B = 0 \qquad H_A = C_1, \, H_B = 0$
- $F_{A,B} = (1/2)\{V_{A,B} + V^*_{A,B}\} \qquad H_{A,B} = V_{A,B}$

Thomson principle

- Reversible
- $$\frac{1}{\mathrm{cap}(A,B)} = \inf_{\Phi} \sum_{\eta\sim\xi} \frac{1}{\mu(\eta)\,R(\eta,\xi)} \Phi(\eta,\xi)^2$$
- Φ unitary flow from A to B:
  - Flow: $\Phi(\eta,\xi) = -\Phi(\xi,\eta)$
  - Divergence free: $\forall\, \eta \notin A \cup B \quad \sum_{\xi\sim\eta} \Phi(\eta,\xi) = 0$
  - Unitary: $\sum_{\eta\in A} \sum_{\xi\notin A} \Phi(\eta,\xi) = 1$

A. Condition (C1): Process visits points

$$\int_0^{t\theta_N} \Big\{ G_{x,y}(\eta_s^{\mathcal{E}}) - \hat{G}_{x,y}(\eta_s^{\mathcal{E}}) \Big\} \, ds \ \longrightarrow 0 \qquad \textbf{(C1)}$$

- $\forall x \qquad \exists \xi^x \in \mathcal{E}_N^x$

$$\lim_{N\to\infty} \inf_{\eta \in \mathcal{E}_N^x} \mathbb{P}_\eta^N \Big[H(\xi^x) < H(\cup_{y\neq x} \mathcal{E}_N^y) \Big] \ = \ 1 \ .$$

A. Condition (C1): Process visits points

$$\int_0^{t\theta_N} \left\{ G_{x,y}(\eta_s^{\mathcal{E}}) - \hat{G}_{x,y}(\eta_s^{\mathcal{E}}) \right\} ds \;\longrightarrow 0 \qquad \textbf{(C1)}$$

- $\forall x \quad \exists \xi^x \in \mathcal{E}_N^x$

$$\lim_{N\to\infty} \inf_{\eta \in \mathcal{E}_N^x} \mathbb{P}_\eta^N \Big[H(\xi^x) < H(\cup_{y\neq x} \mathcal{E}_N^y) \Big] \;=\; 1\,.$$

$$\lim_{N\to\infty} \sup_{\eta \in \mathcal{E}_N^x} \frac{\mathrm{cap}_N\left(\mathcal{E}_N^x, \bigcup_{y\neq x} \mathcal{E}_N^y\right)}{\mathrm{cap}_N(\eta, \xi^x)} \;=\; 0$$

Condition (C1): General case

$$\Big(\mathbb{E}^{\mathcal{E}}_{\nu_N}\Big[\sup_{t\le T}\Big|\int_0^t f(\eta^{\mathcal{E}}(s\,\theta_N))\,ds\Big|\Big]\Big)^2$$

$$\le\ \frac{24\,T}{\theta_N}\,E_{\pi_{\mathcal{E}}}\Big[\Big(\frac{\nu_N}{\pi_{\mathcal{E}}}\Big)^2\Big]\sum_{x\in S}\pi_{\mathcal{E}}(\mathcal{E}^x_N)\,\mathfrak{g}^{-1}_{\mathbf{r},x}\,\langle f,f\rangle_{\pi_x}$$

- ν_N
- $\mathfrak{g}_{\mathbf{r},x}$ spectral gap of reflected process on $\mathcal{E}^x_N$
- $\forall x \quad \sum_{\eta\in\mathcal{E}^x_N}\mu(\eta)\,f(\eta) \ = 0$

$$\frac{\mathfrak{g}^{-1}_{\mathbf{r},x}}{\theta_N}\,E_{\pi_{\mathcal{E}}}\Big[\Big(\frac{\nu_N}{\pi_{\mathcal{E}}}\Big)^2\Big]\ \to\ 0$$

Capacity and mean rates, Condition (C2)

- $$r_{\mathcal{E}_N}(x,y) \;=\; \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta\in\mathcal{E}_N^x} \mu_N(\eta)\, R^{\mathcal{E}_N}(\eta,\mathcal{E}_N^y)$$

$$\theta_N\, r_{\mathcal{E}_N}(x,y) \;\longrightarrow\; r(x,y) \qquad \textbf{(C2)}$$

$$A\subset K\subset E \qquad \mu(A)\, r_K(A, K\setminus A) \;=\; \mathrm{cap}(A, K\setminus A)$$

Capacity and mean rates, Condition (C2)

- Reversible: $A, B \subset K \subset E \quad A \cap B = \varnothing$

$$2\,\mu(A)\,r_K(A,B) \;=\; \mathrm{cap}(A, K \setminus A) + \mathrm{cap}(B, K \setminus B) - \mathrm{cap}(A \cup B, K \setminus [A \cup B])$$

Capacity and mean rates, Condition (C2)

- Reversible: $A, B \subset K \subset E \qquad A \cap B = \varnothing$

$$2\,\mu(A)\,r_K(A,B) \;=\; \mathrm{cap}(A, K\setminus A) + \mathrm{cap}(B, K\setminus B) - \mathrm{cap}(A\cup B, K\setminus[A\cup B])$$

- Non-reversible: $A, B \subset K \subset E \qquad A \cap B = \varnothing$

$$\inf_F \sup_H \Big\{ 2\langle F\,,\, LH\rangle_\mu \;-\; \langle H, (-L)H\rangle_\mu \Big\}$$

- $F_A = 1,\, F_B = C_1,\, F_{K\setminus(A\cup B)} = 0 \qquad H_A = C_2,\, H_B = C_3,$ $H_{K\setminus(A\cup B)} = 0$
- $F^{\mathrm{opt}} \qquad H^{\mathrm{opt}}$

$$H_B^{\mathrm{opt}} \;=\; \frac{r_K(B,A)}{r_K(B, K\setminus B)}$$

Condition (C3)

- Assume asymptotic process has no absorbing points. For all x:

$$\lim_{N\to\infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} = 0$$

Theorem

- Condition (C1)
- $R_N^{\mathcal{E}_N}$ rates of the trace process on $\mathcal{E}_N$

$$r_{\mathcal{E}_N}(x,y) \;=\; \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta)\, R_N^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y)$$

$$\lim_{N\to\infty} \theta_N r_{\mathcal{E}_N}(x,y) \;=\; r(x,y) \qquad \textbf{(H2)}$$

- Assume that process with rates r has no absorbing points

$$\lim_{N\to\infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} \;=\; 0 \qquad \textbf{(H3)}$$

Zero range dynamics

- $$\frac{\ell_N^{1+\alpha(\kappa-1)}}{N^{1+\alpha}} \to 0$$
- $\mathcal{E}_N^x = \{\eta : \eta_x \geq N - \ell_N\}$

Zero range dynamics

- $\dfrac{\ell_N^{1+\alpha(\kappa-1)}}{N^{1+\alpha}} \;\to\; 0$
- $\mathcal{E}_N^x = \{\eta : \eta_x \ge N - \ell_N\}$

- $\displaystyle\inf_{\eta,\xi\in\mathcal{E}_N^x} \mathbf{P}_\eta^N\big[\, H_{\{\xi\}} < H_{\mathcal{E}_N(S\setminus\{x\})} \,\big] \;\to\; 1$
- $X^N_{tN^{\alpha+1}} \to X_t$
- $\displaystyle\mathbf{E}_\eta^N\Big[\int_0^T \mathbf{1}\big\{\eta^N(sN^{\alpha+1}) \in \Delta_N\big\}\, ds\Big] \;\to\; 0\,.$

Zero range dynamics

- $$\frac{\ell_N^{1+\alpha(\kappa-1)}}{N^{1+\alpha}} \to 0$$
- $\mathcal{E}_N^x = \{\eta : \eta_x \geq N - \ell_N\}$

- $$\inf_{\eta,\xi \in \mathcal{E}_N^x} \mathbf{P}_\eta^N \big[H_{\{\xi\}} < H_{\mathcal{E}_N(S\setminus\{x\})} \big] \to 1$$
- $X^N_{tN^{\alpha+1}} \to X_t$
- $$\mathbf{E}_\eta^N \Big[\int_0^T \mathbf{1}\{\eta^N(sN^{\alpha+1}) \in \Delta_N\} \, ds \Big] \to 0 \,.$$

- $$R(x,y) = \frac{\kappa}{\Gamma(\alpha)\, I_\alpha} \operatorname{cap}_S(x,y)$$ uniform measure

Kawasaki dynamics

- $\eta(0) = \eta^{\mathbf{0}}$
- $\xi(t)$ trace of $\eta(t)$ on $\Gamma = \{\eta^{\mathbf{x}} : \mathbf{x} \in \Lambda_L\}$
- $\mathbf{X}(\eta^{\mathbf{x}}) = \mathbf{x}$
- $X(t) = \mathbf{X}(\xi(t))$

Kawasaki dynamics

- $\eta(0) = \eta^{\mathbf{0}}$
- $\xi(t)$ trace of $\eta(t)$ on $\Gamma = \{\eta^{\mathbf{x}} : \mathbf{x} \in \Lambda_L\}$
- $\mathbf{X}(\eta^{\mathbf{x}}) = \mathbf{x}$
- $X(t) = \mathbf{X}(\xi(t))$

- $n^{10}\, L^4\, \{n^3 e^{-\beta} + L e^{-\beta/2}\} \;\to\; 0$
- For every $\delta > 0 \quad n^2\, L^2\, e^{-\delta L/n} \;\to\; 0$
- $Z^\beta(t) = X(tL^2\theta_\beta)/L$
- $Z^\beta(t) \to B(t)$

Kawasaki dynamics

- $c_0 \, \frac{n}{L^2} \, e^{2\beta} \;\leq\; \theta_\beta \;\leq\; C_0 \, n^2 \, e^{2\beta}$

Kawasaki dynamics

- $c_0 \, \dfrac{n}{L^2} \, e^{2\beta} \;\le\; \theta_\beta \;\le\; C_0 \, n^2 \, e^{2\beta}$
- $\Delta = \Omega_{L,K} \setminus \Gamma$
- $n^2 L^2 (L^2 + n^8) \, e^{-\beta} \;\to\; 0$
- $\mathbf{E}^{\beta}_{\eta^{\mathbf{x}}} \Big[\int_0^t \mathbf{1}\{\eta(sL^2\theta_\beta) \in \Delta\} \, ds \Big] \;=\; 0$

Random walks among random traps

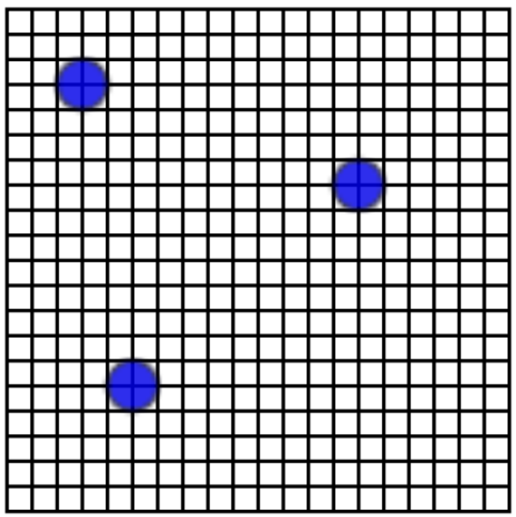
- $\{\mathbb{X}_n : n \geq 1\}$ embedded discrete time chain
- $\mathbb{X}_n$ random walk on G_N
- Stationary state π^N $\quad \pi^N(x) \sim \deg(x)$
- $t_{\text{mix}} \ll$ hitting times

Random walks among random traps

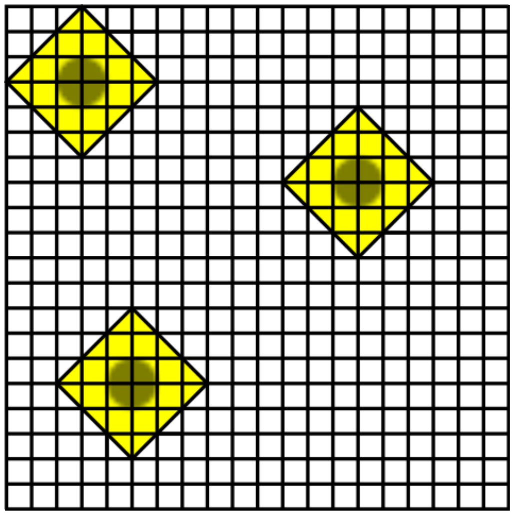
- $\{\mathbb{X}_n : n \geq 1\}$ embedded discrete time chain
- $\mathbb{X}_n$ random walk on G_N
- Stationary state $\pi^N \qquad \pi^N(x) \sim \deg(x)$
- $t_{\text{mix}} \ll$ hitting times
- $\|\mu - \nu\|_{TV} = \dfrac{1}{2} \displaystyle\sum_{x \in V} |\mu(x) - \nu(x)|$
- $t_{\text{mix}} = \min \left\{ n : \max_{x \in V} \|P_n(x, \cdot\,) - \pi(\,\cdot\,)\|_{TV} \leq \dfrac{1}{4} \right\}$
- $\mathbb{H}_B \;=\; \inf\{n \geq 0 : \mathbb{X}_n \in B\} \qquad B \subset V_N$
- $\mathbb{H}_B^+ \;=\; \inf\{n \geq 1 : \mathbb{X}_n \in B\}$
- $\mathbb{T}_N^d \qquad t_{\text{mix}} = O(N^2) \qquad \mathbb{H}_x = O(N^d) \qquad d \geq 3$

- $\{W_j : j \geq 1\} \quad W_j \geq 0 \quad W_1 \geq W_2 \geq \cdots \quad \sum_j W_j < \infty$
- $x_1^N, x_2^N, \ldots, x_{|V_N|}^N$ random permutation of V_N
- $W_{x_j^N}^N = W_j$

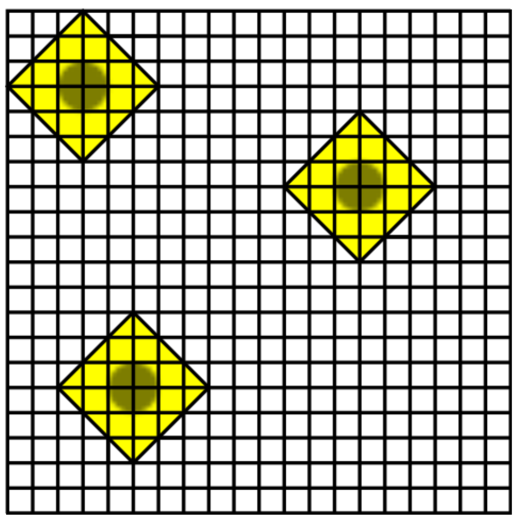
- $\{W_j : j \geq 1\} \quad W_j \geq 0 \quad W_1 \geq W_2 \geq \cdots \quad \sum_j W_j < \infty$
- $x_1^N, x_2^N, \ldots, x_{|V_N|}^N$ random permutation of V_N
- $W_{x_j^N}^N = W_j$
- $M_N \uparrow \infty \qquad A_N = \{x_1^N, \ldots, x_{M_N}^N\}$



- $M_N \uparrow \infty \quad A_N = \{x_1, \dots, x_{M_N}\}$
- $\ell_N \uparrow \infty \quad B(x_j, \ell_N)$



- $M_N \uparrow \infty \qquad A_N = \{x_1, \ldots, x_{M_N}\}$
- $\ell_N \uparrow \infty \qquad B(x_j, \ell_N)$
- $B(x_j, \ell_N)$ do not overlap
- $z \notin \bigcup_j B(x_j, \ell_N) \qquad \mathbb{X}_n$ mix before hitting A_N



- $v_{\ell_N}(x) = \mathbf{P}_x\big[\mathbb{H}(B(x,\ell_N)^c) < \mathbb{H}_x^+\big]$ escape probability $x \in A_N$
- $\mathfrak{N}(x) = \#\{\mathbb{X}_n$ visits x before escaping $\}$ $\mathfrak{N}(x) \geq 1$
- $\mathfrak{N}(x) \sim$ geometric $\mathbf{P}_x[\mathfrak{N}(x) = 1] = v_{\ell_N}(x)$
- $\displaystyle\int_0^{H(B(x,\ell_N)^c)} \mathbf{1}\{X_s = x\}\, ds$ mean $W_x / v_{\ell_N}(x)$ exponential

- $v_{\ell_N}(x) = \mathbf{P}_x\big[\mathbb{H}(B(x,\ell_N)^c) < \mathbb{H}_x^+\big]$ escape probability $x \in A_N$
- $\mathfrak{N}(x) = \#\{\mathbb{X}_n$ visits x before escaping $\}$ $\mathfrak{N}(x) \geq 1$
- $\mathfrak{N}(x) \sim$ geometric $\mathbf{P}_x[\mathfrak{N}(x) = 1] = v_{\ell_N}(x)$
- $\displaystyle\int_0^{H(B(x,\ell_N)^c)} \mathbf{1}\{X_s = x\}\, ds$ mean $W_x / v_{\ell_N}(x)$ exponential
- $q_N(x_j) = \mathbf{P}_\pi\big[\mathbb{H}(A_N) = \mathbb{H}_{x_j}\big]$
- $\mathbf{P}_z\big[\mathbb{H}(A_N) = \mathbb{H}_{x_j}\big] \sim q_N(x_j) \quad z \notin \displaystyle\bigcup_i B(x_i, \ell_N)$
- $q_N(x) \sim \deg(x)\, v_{\ell_N}(x)$

- $v_{\ell_N}(x) = \mathbf{P}_x\big[\mathbb{H}(B(x,\ell_N)^c) < \mathbb{H}_x^+\big]$ escape probability $x \in A_N$
- $\mathfrak{N}(x) = \#\{\mathbb{X}_n$ visits x before escaping $\}$ $\mathfrak{N}(x) \geq 1$
- $\mathfrak{N}(x) \sim$ geometric $\mathbf{P}_x[\mathfrak{N}(x) = 1] = v_{\ell_N}(x)$
- $\displaystyle\int_0^{H(B(x,\ell_N)^c)} \mathbf{1}\{X_s = x\}\, ds$ mean $W_x / v_{\ell_N}(x)$ exponential
- $q_N(x_j) = \mathbf{P}_\pi\big[\mathbb{H}(A_N) = \mathbb{H}_{x_j}\big]$
- $\mathbf{P}_z\big[\mathbb{H}(A_N) = \mathbb{H}_{x_j}\big] \sim q_N(x_j)$ $\displaystyle z \notin \bigcup_i B(x_i, \ell_N)$
- $q_N(x) \sim \deg(x)\, v_{\ell_N}(x)$
- Pseudo-transitive: $\mathbb{P}\big[(\mathfrak{x}, B(\mathfrak{x}, \ell_N)) \not\equiv (\mathfrak{y}, B(\mathfrak{y}, \ell_N))\big] \;\to\; 0$

- $\Psi_N : V_N \to \mathbb{N} \quad \Psi_N(x_j^N) = j$
- $\Psi_N(X_t^N)$ Markov process on $\{1, \ldots, |V_N|\}$
- K_t^N trace of $\Psi_N(X_t^N)$ on $\{1, \ldots, M_N\}$

- $\Psi_N : V_N \to \mathbb{N} \quad \Psi_N(x_j^N) = j$
- $\Psi_N(X_t^N)$ Markov process on $\{1, \dots, |V_N|\}$
- K_t^N trace of $\Psi_N(X_t^N)$ on $\{1, \dots, M_N\}$

- Mean $W_k / v_{\ell_N}(x_k) \sim W_k / v_{\ell_N}(x_1)$ exponential times
- $p_N(i,j) \sim \mathbf{P}_\pi\big[\mathbb{H}(A_N) = \mathbb{H}_{x_j}\big] \;=\; q_N(x_j) \sim \deg(x_j)\, v_{\ell_N}(x_j)$
- $p_N(i,j) \sim 1/M_N$

- $\Psi_N : V_N \to \mathbb{N} \quad \Psi_N(x_j^N) = j$
- $\Psi_N(X_t^N)$ Markov process on $\{1, \dots, |V_N|\}$
- K_t^N trace of $\Psi_N(X_t^N)$ on $\{1, \dots, M_N\}$

- Mean $W_k/v_{\ell_N}(x_k) \sim W_k/v_{\ell_N}(x_1)$ exponential times
- $p_N(i,j) \sim \mathbf{P}_\pi\left[\mathbb{H}(A_N) = \mathbb{H}_{x_j}\right] \;=\; q_N(x_j) \sim \deg(x_j)\, v_{\ell_N}(x_j)$
- $p_N(i,j) \sim 1/M_N$

- $\beta_N = v_{\ell_N}(x_1)^{-1}$
- $K_{t\beta_N}^N$ converges K-process $(W_1, W_2, \dots), (1, 1, \dots)$
- $\displaystyle\int_0^T \mathbf{1}\{X_{s\beta_N} \notin A_N\}\, ds \longrightarrow 0$

K processes

- Markov process on $\mathbb{N} \cup \{\infty\}$
- $\{u_k : k \geq 1\}$ entrance measure
- $\{Z_k : k \geq 1\}$ mean exponential times
- $\sum_{k\geq 1} u_k\, Z_k < \infty \qquad \sum_{k\geq 1} u_k = \infty$

Fontes-Mathieu 2008 $u_k = 1$

K processes

- Markov process on $\mathbb{N} \cup \{\infty\}$
- $\{u_k : k \geq 1\}$ entrance measure
- $\{Z_k : k \geq 1\}$ mean exponential times
- $\sum_{k\geq 1} u_k\, Z_k < \infty \qquad \sum_{k\geq 1} u_k = \infty$

Fontes-Mathieu 2008 $\quad u_k = 1$

- Mean Z_k exponential time
- Jumps to ∞ Immediately returns to $\mathbb{N}$
- A finite $\quad P[H_A = H_k] = \frac{u_k}{\sum_{j\in A} u_j}$
- $u_k \longrightarrow \gamma\, u_k$