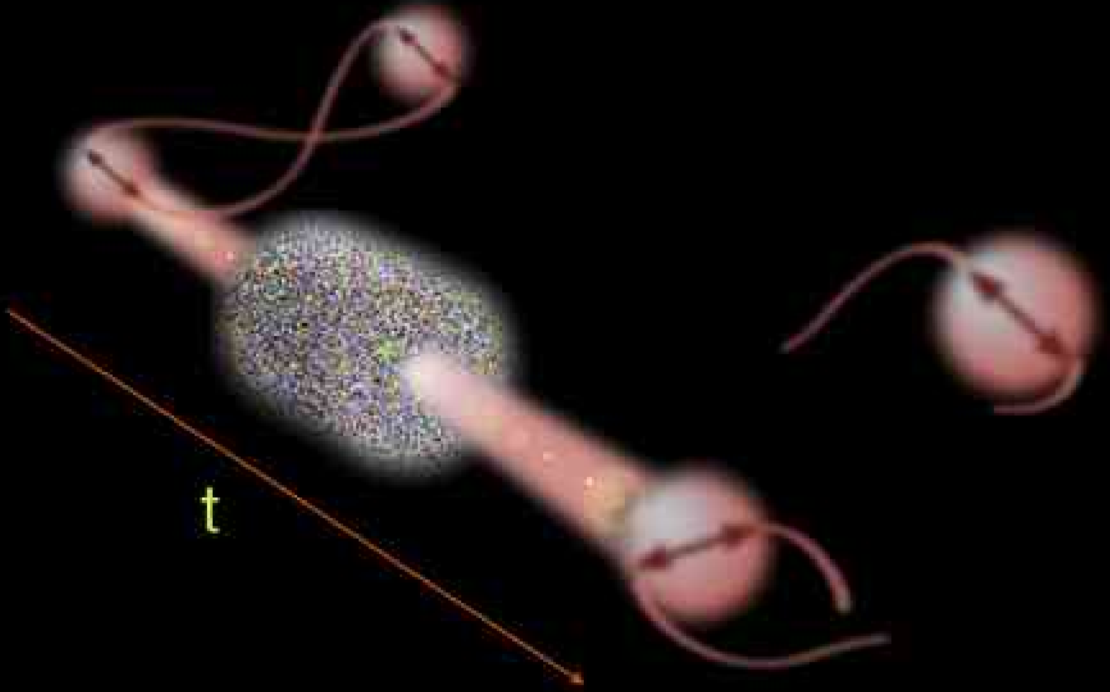
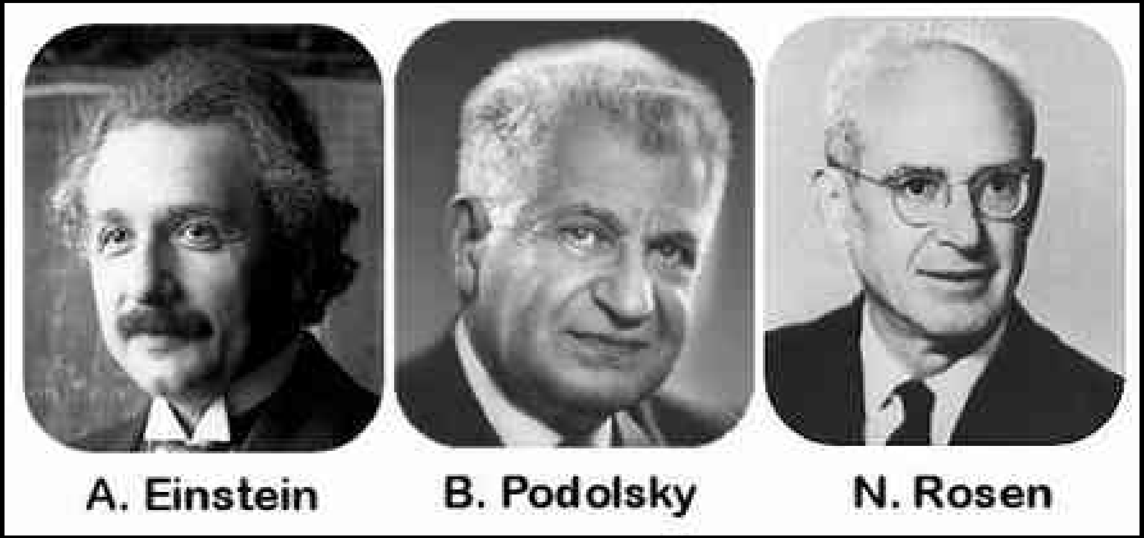
Open system dynamics of entanglement



Fernando de Melo
qig@CBPF

Entanglement...



Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777 (1935)

… the way of “paradoxes”

Best possible knowledge of a whole does not include best possible knowledge of its parts — and that is what keeps coming back to haunt us

Erwin Schrödinger (1935)

Entanglement...

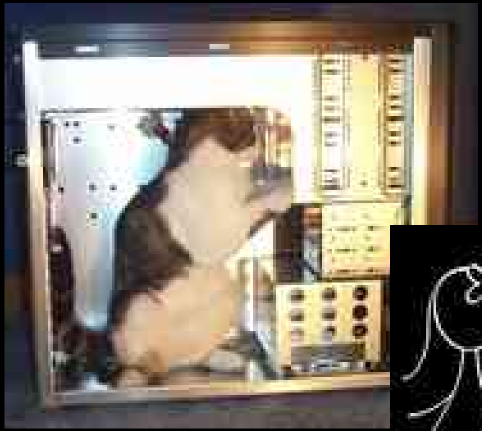
“On the Einstein-Podolsky-Rosen Paradox”
Physics 1, 195 (1964)

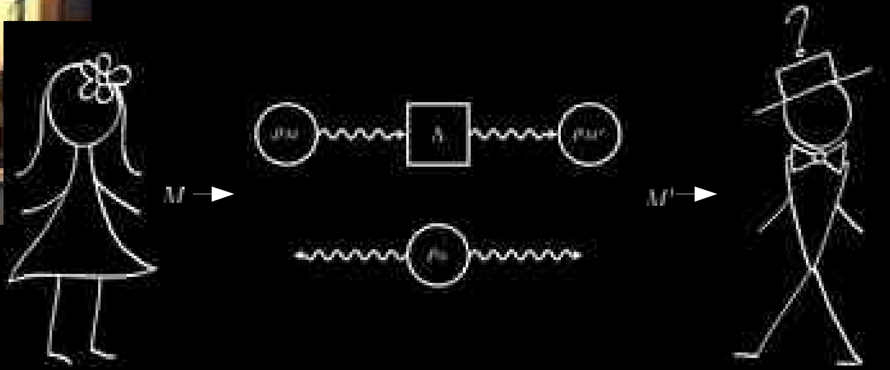
... correlation with physical consequences

Entanglement...

… a resource for quantum information

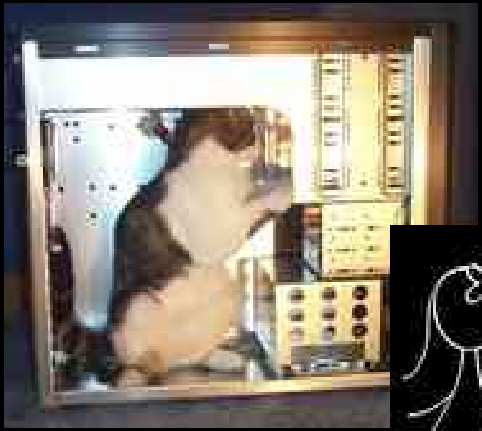
Entanglement...

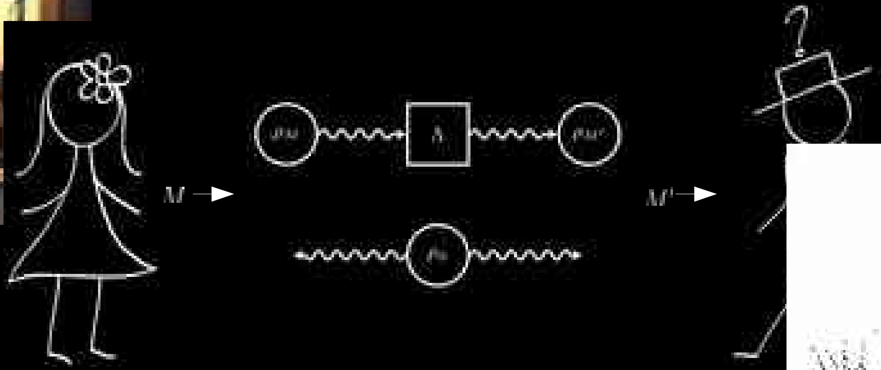


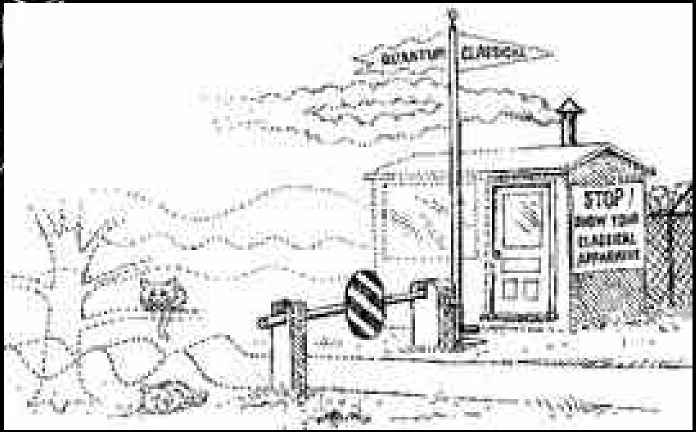


... a resource for quantum information

Entanglement…

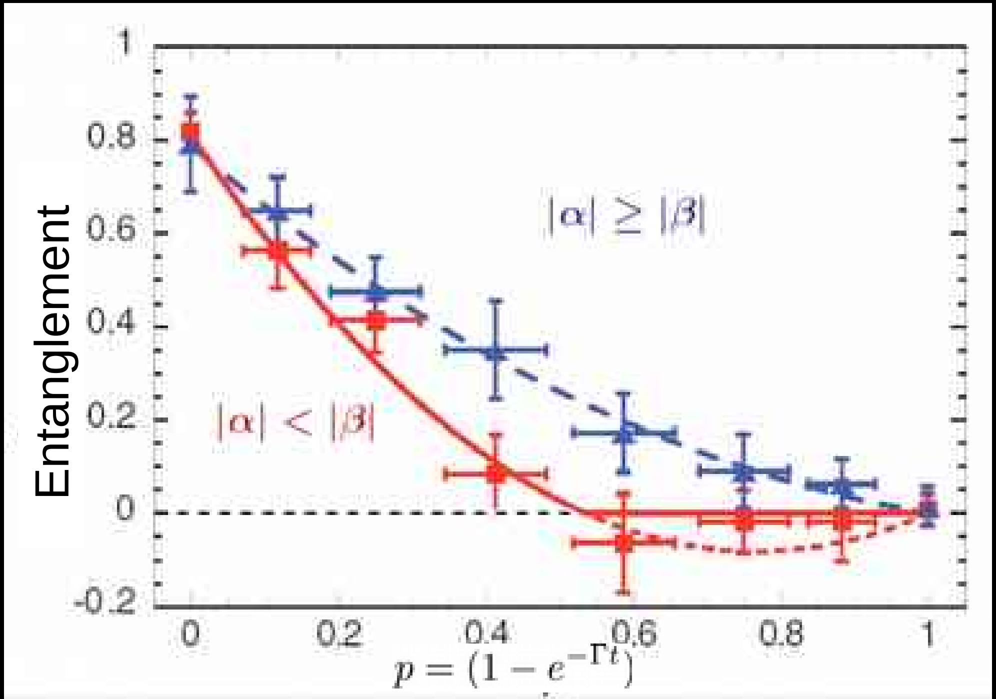






… a resource for quantum information

Caveat: Entanglement is fragile



M.P Almeida, FdM, M. Hor-Meyll, A. Salles, S.P. Walborn, P.H. Ribeiro, L. Davidovich Science 316, 579 (2007)

Entanglement open system dynamics

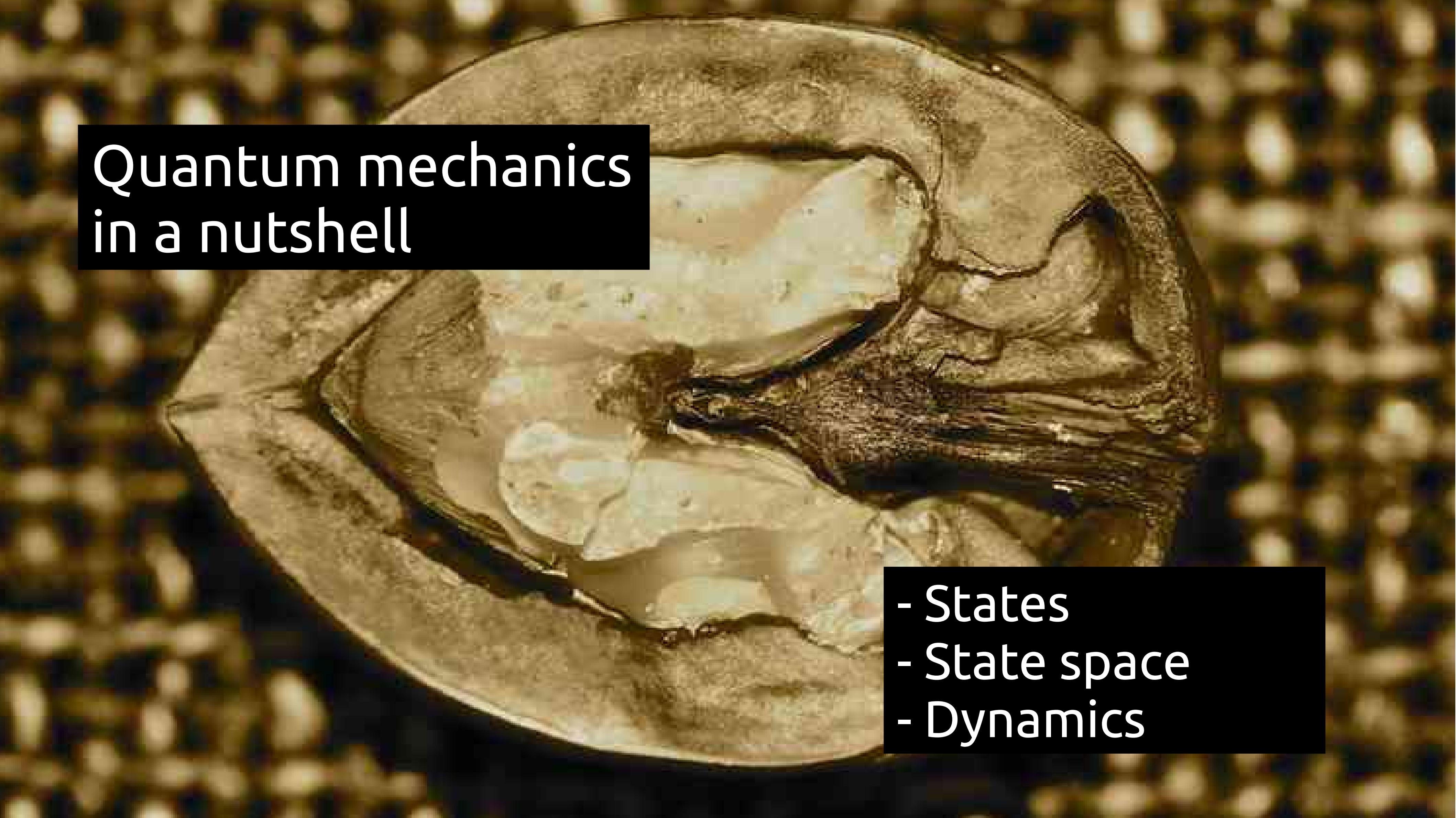
$$
\begin{tikzcd}
\varrho_0 \arrow[r, "\Lambda_t"] \arrow[d] & \varrho_t \arrow[d] \\
E(\varrho_0) & E(\varrho_t)
\end{tikzcd}
$$

Entanglement open system dynamics

$$\begin{tikzcd}
\varrho_0 \arrow[r, "\Lambda_t"] \arrow[d] & \varrho_t \arrow[d] \\
E(\varrho_0) \arrow[r, Rightarrow] & E(\varrho_t)
\end{tikzcd}$$

Outline

- Quantum mechanics in a nutshell
- Entanglement: definition and quantification
- Entanglement dynamics
  - Deterministic equation of motion
  - Statistical approach: a universal behavior
- Conclusions and open questions



Quantum mechanics in a nutshell

- States
- State space
- Dynamics

1st postulate: To every physical system is associated a Hilbert space $\mathscr{H}$. The state of a quantum system is described by a unit vector $|\psi\rangle \in \mathscr{H}$.

Given an orthonormal basis $\{|e_i\rangle\}_{i=0}^{d-1} \in \mathbb{C}^d$ then

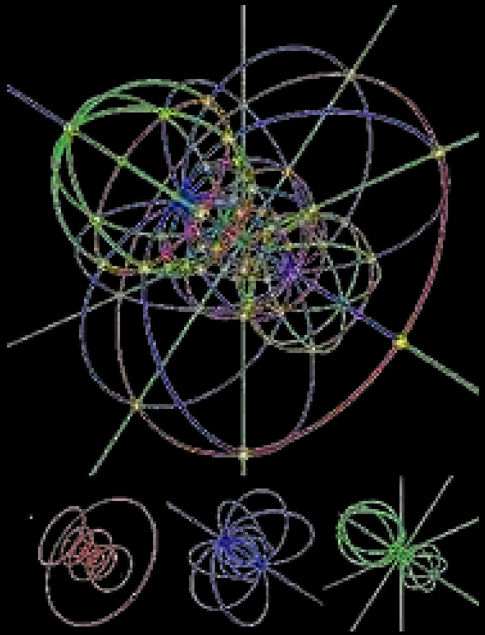
$$|\psi\rangle = \sum_{i=0}^{d-1} c_i |e_i\rangle$$

Constraints:

Normalization: $$\sum_{i=0}^{d-1} |c_i|^2 = 1$$

Modulo global phase: $c_0 \in \mathbb{R}$

State space of $|\psi\rangle$'s : 2d-2 sphere



Consider the case where the state of the system is not exactly know, but it is in the state $|\psi_i\rangle$ with probability p_i .

The expectation value of an observable $\mathcal{A}$ (Hermitian linear operator), is given by:

$$\langle \mathcal{A} \rangle = \sum_i p_i \langle \psi_i | \left(\mathcal{A} \, | \psi_i \rangle \right)$$

The expectation value of an observable $\mathcal{A}$ (Hermitian linear operator), is given by:

$$\begin{aligned}\langle\mathcal{A}\rangle &= \sum_i p_i \langle\psi_i|\left(\mathcal{A}|\psi_i\rangle\right) \\ &= \mathrm{Tr}\Bigg[\underbrace{\left(\sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|\right)}_{\varrho}\mathcal{A}\Bigg]\end{aligned}$$

The expectation value of an observable $\mathcal{A}$ (Hermitian linear operator), is given by:

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\langle \mathcal{A} \rangle &= \sum_i p_i \langle \psi_i | \left(\mathcal{A} \, | \psi_i \rangle \right) \\
&= \operatorname{Tr} \Bigg[\underbrace{\left(\sum_i p_i \, | \psi_i \rangle \, \langle \psi_i | \right)}_{\varrho} \mathcal{A} \Bigg] \\
&= \operatorname{Tr} \varrho \mathcal{A}
\end{aligned}$$

Properties of the density matrix:

- linear operator $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
- trace one $\mathrm{Tr}\,\varrho = \sum_i p_i \,\mathrm{Tr}\,|\psi_i\rangle\langle\psi_i| = \sum_i p_i = 1$
- Hermitian $\varrho^\dagger = \sum_i p_i^* (|\psi\rangle\langle\psi|)^\dagger = \varrho$
- positive semi-definite $\forall\, |\chi\rangle \in \mathscr{H}\;\; \langle\chi|\varrho\,|\chi\rangle \geq 0$

1st postulate reloaded: To every physical system is associated a Hilbert space $\mathscr{H}$. The state of a quantum system is described by a density matrix $\varrho \in \mathcal{D}(\mathscr{H})$

The previous postulate is then a special case with the identification $\varrho_\psi = |\psi\rangle\langle\psi| \rightarrow |\psi\rangle$

State space of ϱ's : convex set

If ϱ_1 and ϱ_2 are density matrices, then

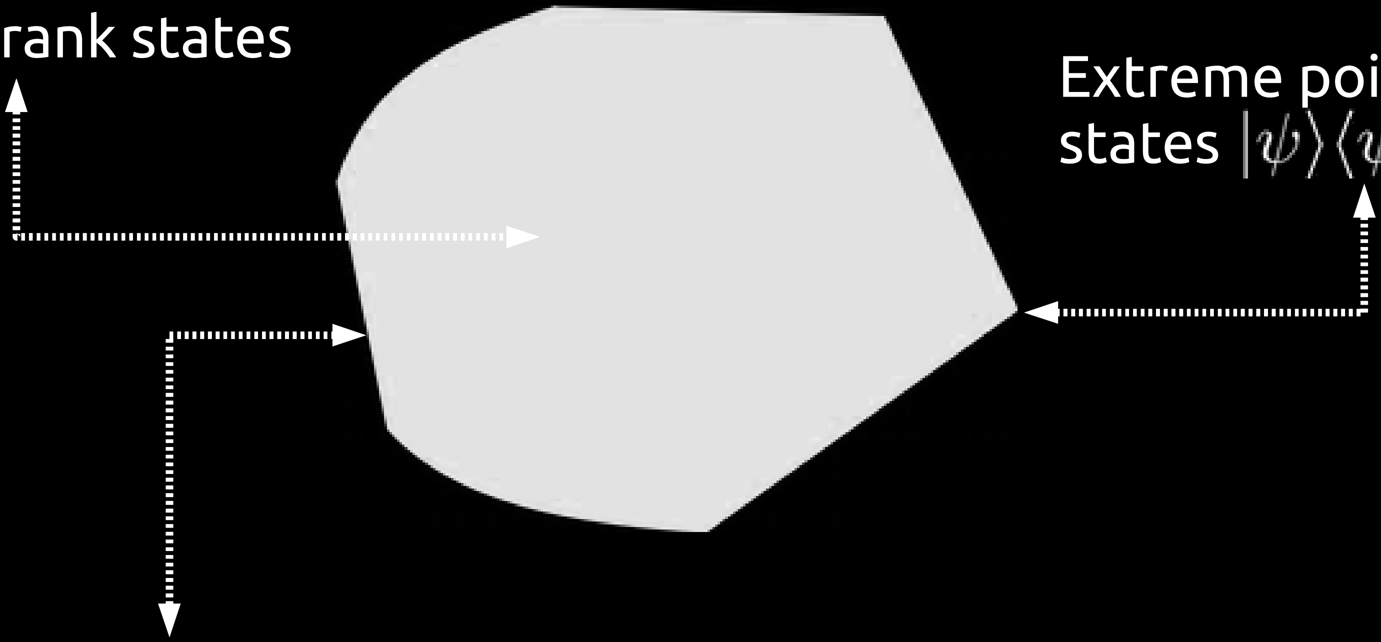
$$\lambda\varrho_1 + (1-\lambda)\varrho_2$$

with $\lambda \in [0,1]$, is also a density matrix.

State space of ϱ's : convex set

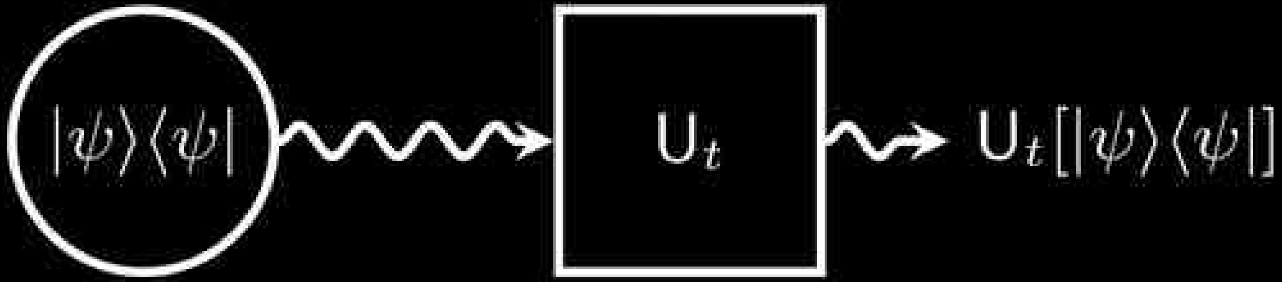
Inside: full rank states

Extreme points: pure states $|\psi\rangle\langle\psi|$



At the facets: non-full rank states

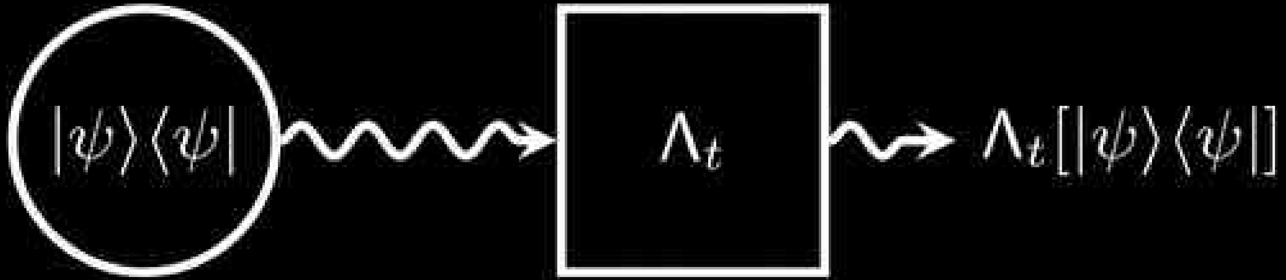
Dynamics: Closed system / Noiseless



Where $\mathsf{U}_t[|\psi\rangle\langle\psi|] := U_t|\psi\rangle\langle\psi|U_t^\dagger$, and $U_t^\dagger U_t = \mathbb{1}$

Pure states remain pure – no information is lost

Dynamics: Open system / Noisy



Where $\Lambda_t[|\psi\rangle\langle\psi|] := \sum_i K_t^{(i)} |\psi\rangle\langle\psi| {K_t^{(i)}}^\dagger$ and $\sum_i {K_t^{(i)}}^\dagger K_t^{(i)} = \mathbb{1}$

Pure states might become mixed (not rank 1) – information might be lost

Dynamics: Open system / Noisy

The linear map $\Lambda_t : \mathcal{D}(\mathscr{H}) \mapsto \mathcal{D}(\mathscr{H})$

- preserves the trace

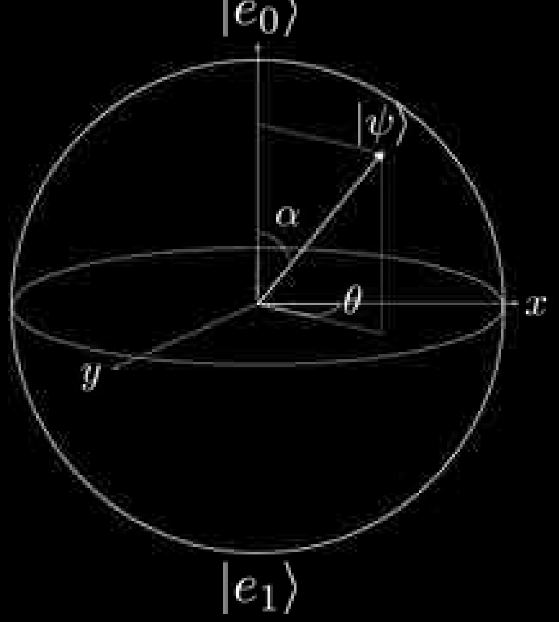
- preserves Hermiticity

- preserves positive semi-definiteness
(more than that, it is completely positive)

Example: 1 qubit dephasing

Consider the state,

$$|\psi\rangle = \cos\frac{\alpha}{2}|e_0\rangle + e^{i\theta}\sin\frac{\alpha}{2}|e_1\rangle$$



Its density matrix reads

$$\varrho_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\frac{\alpha}{2} & e^{-i\theta}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\ e^{i\theta}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}$$

Example: 1 qubit dephasing

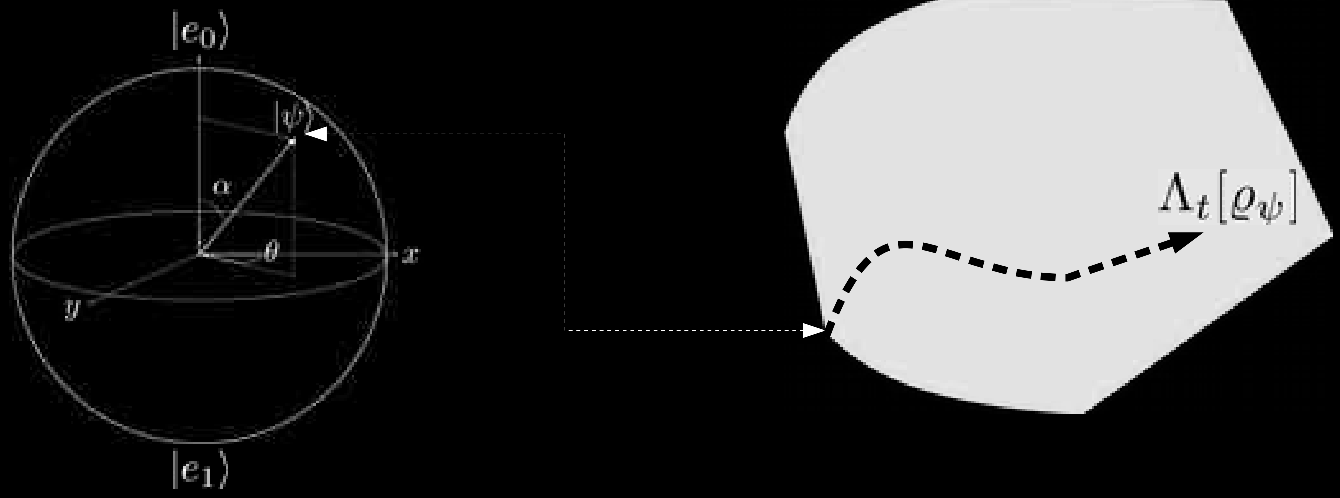
Now consider the map Λ_t with

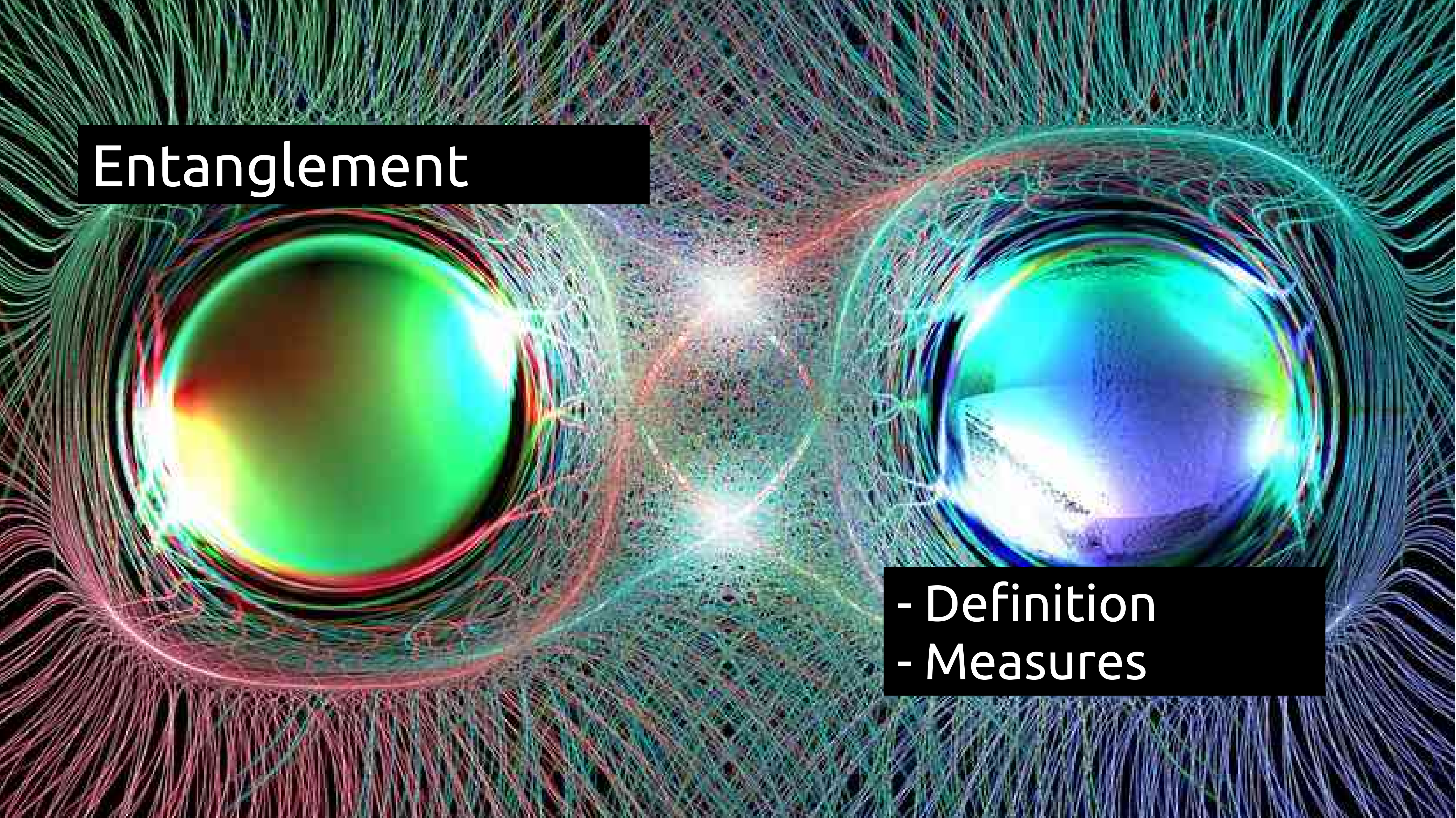
$$K_t^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\Gamma t/2} \end{pmatrix} \qquad K_t^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - e^{-\Gamma t/2}} \end{pmatrix}$$

with $\Gamma > 0$ is a decay rate

The dynamics is then

$$\Lambda_t[\varrho_\psi] = \sum_i K_t^{(i)} |\psi\rangle\langle\psi| K_t^{(i)\dagger} = \begin{pmatrix} \cos^2\frac{\alpha}{2} & e^{-\Gamma t/2} e^{-i\theta} \sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\ e^{-\Gamma t/2} e^{i\theta} \sin\frac{\alpha}{2}\cos\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}$$





Entanglement

- Definition
- Measures

Consider now a system composed of two subsystems, say A and B. It is assigned to it a composed Hilbert space $\mathscr{H} = \mathscr{H}_{\mathrm{A}} \otimes \mathscr{H}_{\mathrm{B}}$

But there exist states

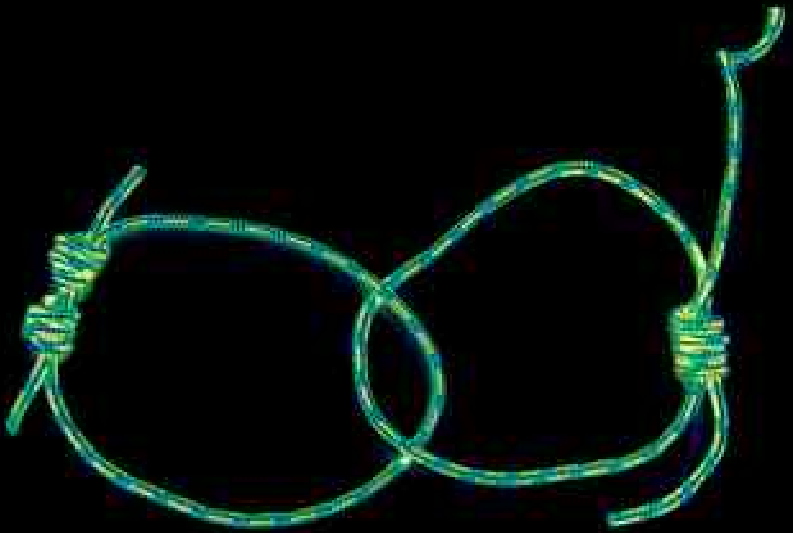
$$|\psi\rangle \in \mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$$

for which there exist **NO** $|\phi\rangle \in \mathscr{H}_A$ and $|\chi\rangle \in \mathscr{H}_B$, such that

$$|\psi\rangle \stackrel{!}{=} |\phi\rangle \otimes |\chi\rangle$$

These are the entangled (pure) states

Bipartite entangled state



$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e_0\rangle \otimes |e_0\rangle + |e_1\rangle \otimes |e_1\rangle)$$

Tripartite entangled states

Cluster state

$$\frac{1}{2}\left(|e_0e_0e_0\rangle + |e_0e_0e_1\rangle + |e_0e_1e_0\rangle - |e_0e_1e_1\rangle + \right.$$
$$\left. + |e_1e_0e_0\rangle + |e_1e_0e_1\rangle - |e_1e_1e_0\rangle + |e_1e_1e_1\rangle\right)$$

GHZ state

$$\frac{1}{\sqrt{2}}\left(|e_0e_0e_0\rangle + |e_1e_1e_1\rangle\right)$$

Entanglement definition

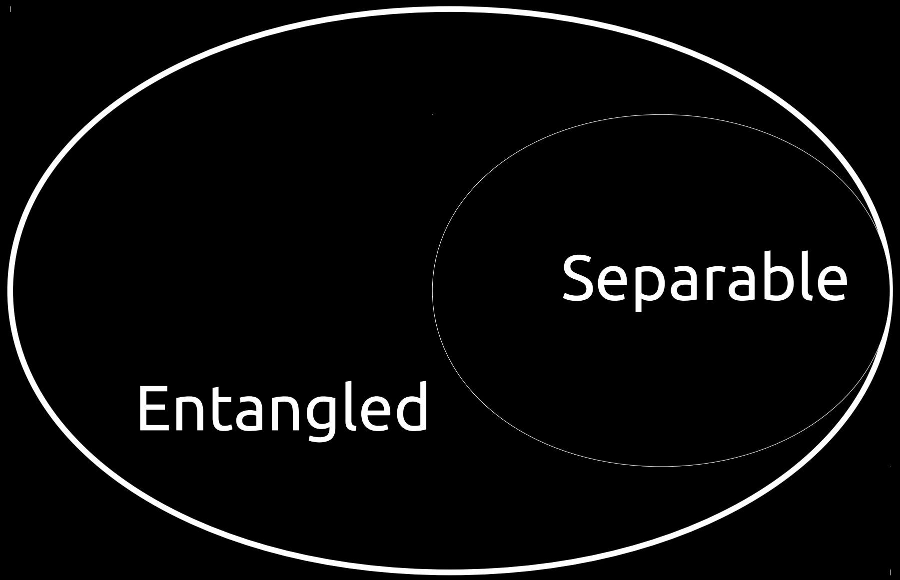
A multipartite state $\varrho \in \mathcal{D}(\mathscr{H}_A) \otimes \mathcal{D}(\mathscr{H}_B) \otimes \cdots \otimes \mathcal{D}(\mathscr{H}_N)$ is **separable** if it can be written as

$$\varrho = \sum_i p_i |\psi_A^{(i)}\rangle\langle\psi_A^{(i)}| \otimes |\psi_B^{(i)}\rangle\langle\psi_B^{(i)}| \otimes \cdots \otimes |\psi_N^{(i)}\rangle\langle\psi_N^{(i)}|$$

with $p_i \geq 0$ and $\sum_i p_i = 1$

The state is **entangled** otherwise

Entanglement in the space of states



Watch out: This is not a Venn diagram!

Entanglement measures

Let $E : \mathcal{D}(\mathscr{H}_A) \otimes \cdots \otimes \mathcal{D}(\mathscr{H}_N) \mapsto [0,1]$ be a tentative entanglement measure. It then must satisfy:

- $E(\varrho) = 0$ iff ϱ is separable
- $E(\lambda\varrho + (1-\lambda)\varrho') \leq \lambda E(\varrho) + (1-\lambda)E(\varrho'), \ \forall\lambda \in [0,1]$
- $|E(\varrho) - E(\varrho')| \xrightarrow{||\varrho-\varrho'||\to 0} 0$
- $E(\Lambda_{\mathrm{LOCC}}[\varrho]) \leq E(\varrho)$

Entanglement measures

Recipe to construct an entanglement measure:

1. Take a valid measure over pure states

2. Extend it over density matrices. How?

Entanglement measures

Watch out: Entanglement measures cannot be linear!

Consider a purported measure E and the entangled states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|e_0 e_0\rangle + |e_1 e_1\rangle) \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|e_0 e_0\rangle - |e_1 e_1\rangle)$$

s.t. $E(|\Phi^+\rangle\langle\Phi^+|) = E_+ > 0$ and $E(|\Phi^-\rangle\langle\Phi^-|) = E_- > 0$

Entanglement measures

Watch out: Entanglement measures cannot be linear!

If linear, then

$$E\left(\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|\right) = \frac{1}{2}(E_+ + E_-) > 0,$$

Despite of the fact that

$$\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}\left(|e_0\rangle\langle e_0| \otimes |e_0\rangle\langle e_0| + |e_1\rangle\langle e_1| \otimes |e_1\rangle\langle e_1|\right)$$

is separable!

Entanglement measures

Given the state

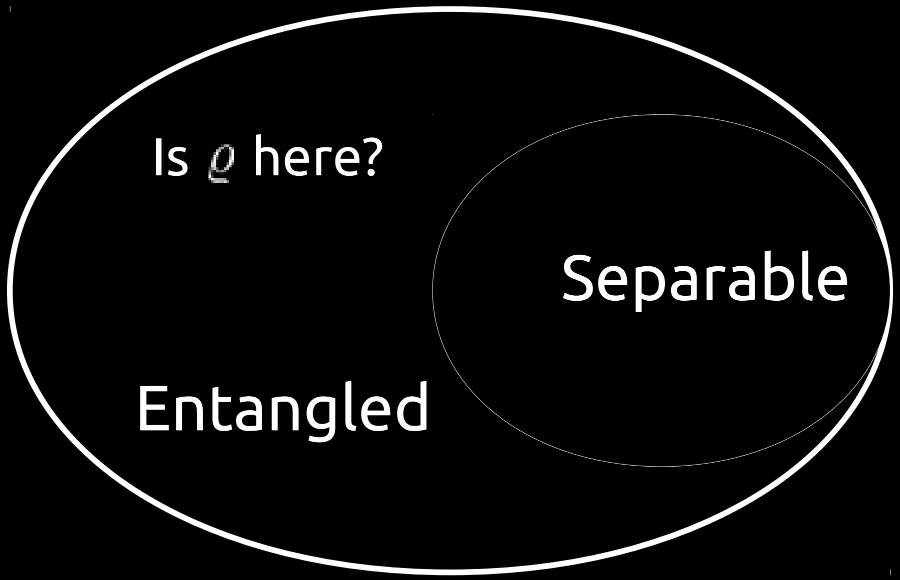
$$\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

then

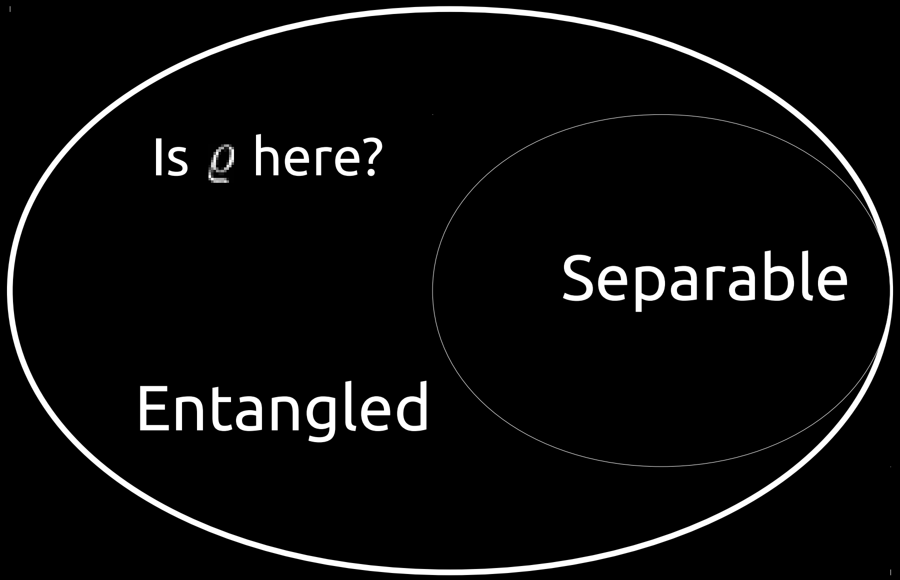
$$E(\varrho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle)$$

A. Uhlmann, PRA 62, 032307 (2000)

Finally: given ϱ, is it entangled and how strongly?



Finally: given ϱ, is it entangled and how strongly?

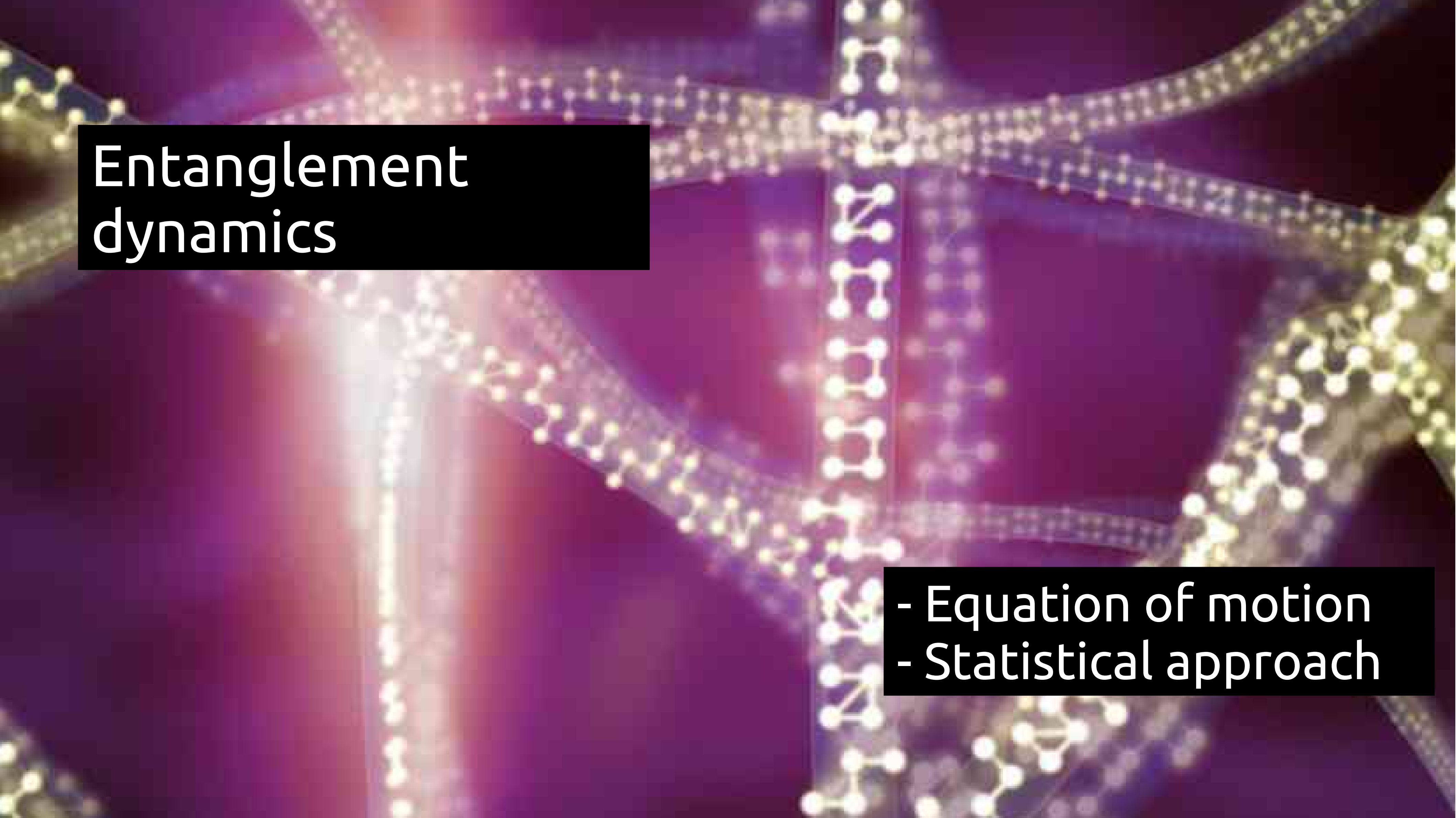


It's NP-Hard to discover that!

Only known cases:
2x2 and 2x3

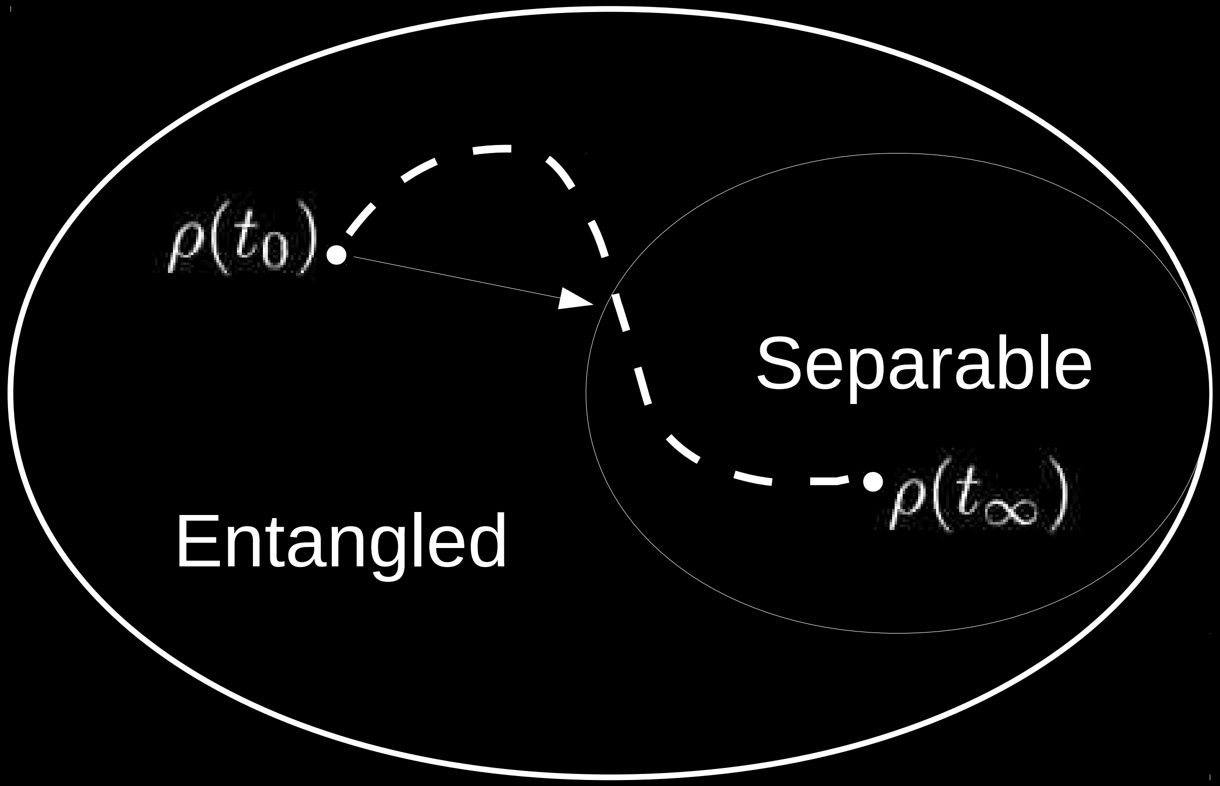
L. Gurvits,
In Proceedings of the thirty-fifth ACM symposium on Theory of computing, pages 10 –19, (2003), ACM Press

Entanglement dynamics



- Equation of motion
- Statistical approach

Where are we?



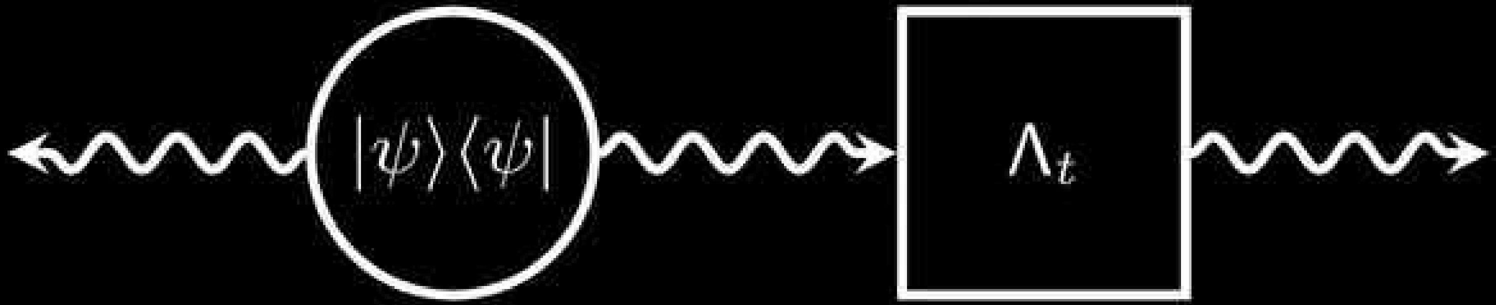
In trouble, definitely.

Dynamical approach:

$$\begin{tikzcd}
\varrho_0 \arrow[r, "\Lambda_t"] \arrow[d] & \varrho_t \arrow[d] \\
E(\varrho_0) \arrow[r, Rightarrow] & E(\varrho_t)
\end{tikzcd}$$

Deterministic equation of motion

Bipartite states (dxd)



$$C_d\Big(\mathbb{1}\otimes\Lambda_t[|\psi\rangle\langle\psi|]\Big) = C_d\Big(|\psi\rangle\langle\psi|\Big)\; C_d\Big(\mathbb{1}\otimes\Lambda_t[|\Phi_d^+\rangle\langle\Phi_d^+|]\Big)$$

where $|\Phi_d^+\rangle = \dfrac{1}{\sqrt{d}}\displaystyle\sum_{i=0}^{d-1}|e_i\, e_i\rangle$

T. Konrad, FdM, M. Tiersch, C. Kasztelan, A. Aragão, and A. Buchleitner,
Nat. Phys. 4, 99 (2008)
M. Tiersch, FdM, and A. Buchleitner,
Phys. Rev. Lett. 101, 170502, (2008)

Proof ingredients

- (G-)Concurrence: determinant structure

$$|\psi\rangle = \sum_{i,j=0}^{d-1} \psi_{ij}\, |e_i\, e_j\rangle \;\Rightarrow\; C_d(|\psi\rangle) = d|\det \psi|^{2/d}$$

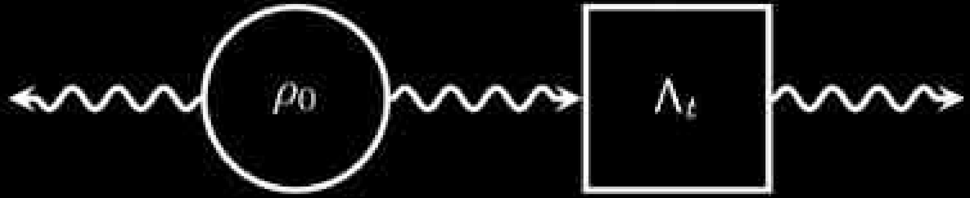
- Jamiołkowsky isomorphism

$$|\psi\rangle = M_\psi \otimes \mathbb{1} \left|\Phi_d^+\right\rangle \quad \text{with} \quad M_\psi = \sqrt{2} \sum_{i,j} \psi_{i,j}\, |e_i\rangle\langle e_j|$$

A. Jamiołkowsky, Rep. Math. Phys. **3**, 275 (1972)

Further results:

Initially mixed states:



$$C_d\Big(\mathbb{1}\otimes\Lambda_t[\varrho_0]\Big)\le C_d(\varrho_0)\;C_d\Big(\mathbb{1}\otimes\Lambda_t[|\Phi_d^+\rangle\langle\Phi_d^+|]\Big)$$

Two-sided channels:

$$C_d\Big(\Phi_t\otimes\Lambda_t[\varrho_0]\Big)\le C_d(\varrho_0)\;C_d\Big(\mathbb{1}\otimes\Lambda_t[|\Phi_d^+\rangle\langle\Phi_d^+|]\Big)C_d\Big(\Phi_t\otimes\mathbb{1}[|\Phi_d^+\rangle\langle\Phi_d^+|]\Big)$$

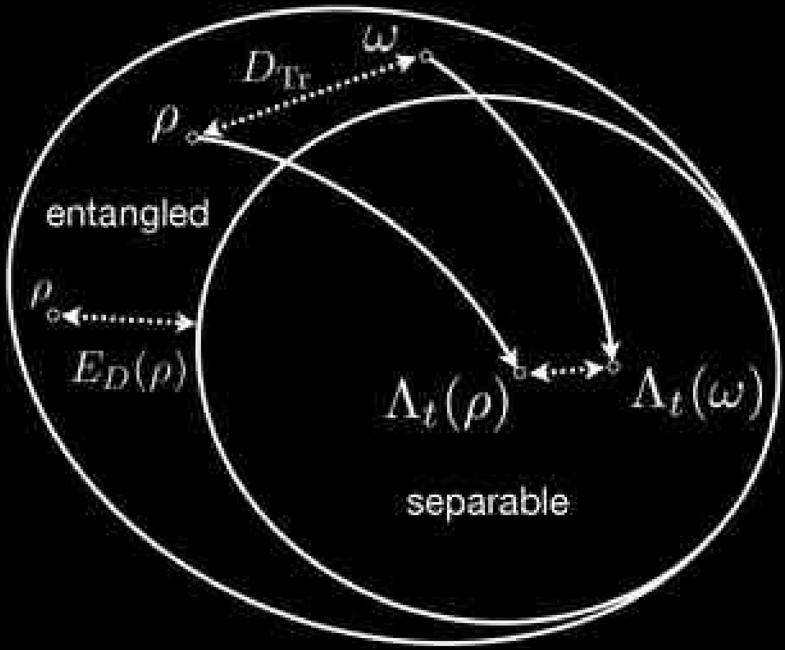
Statistical approach: a universal behavior

Multipartite states

Concentration of measure

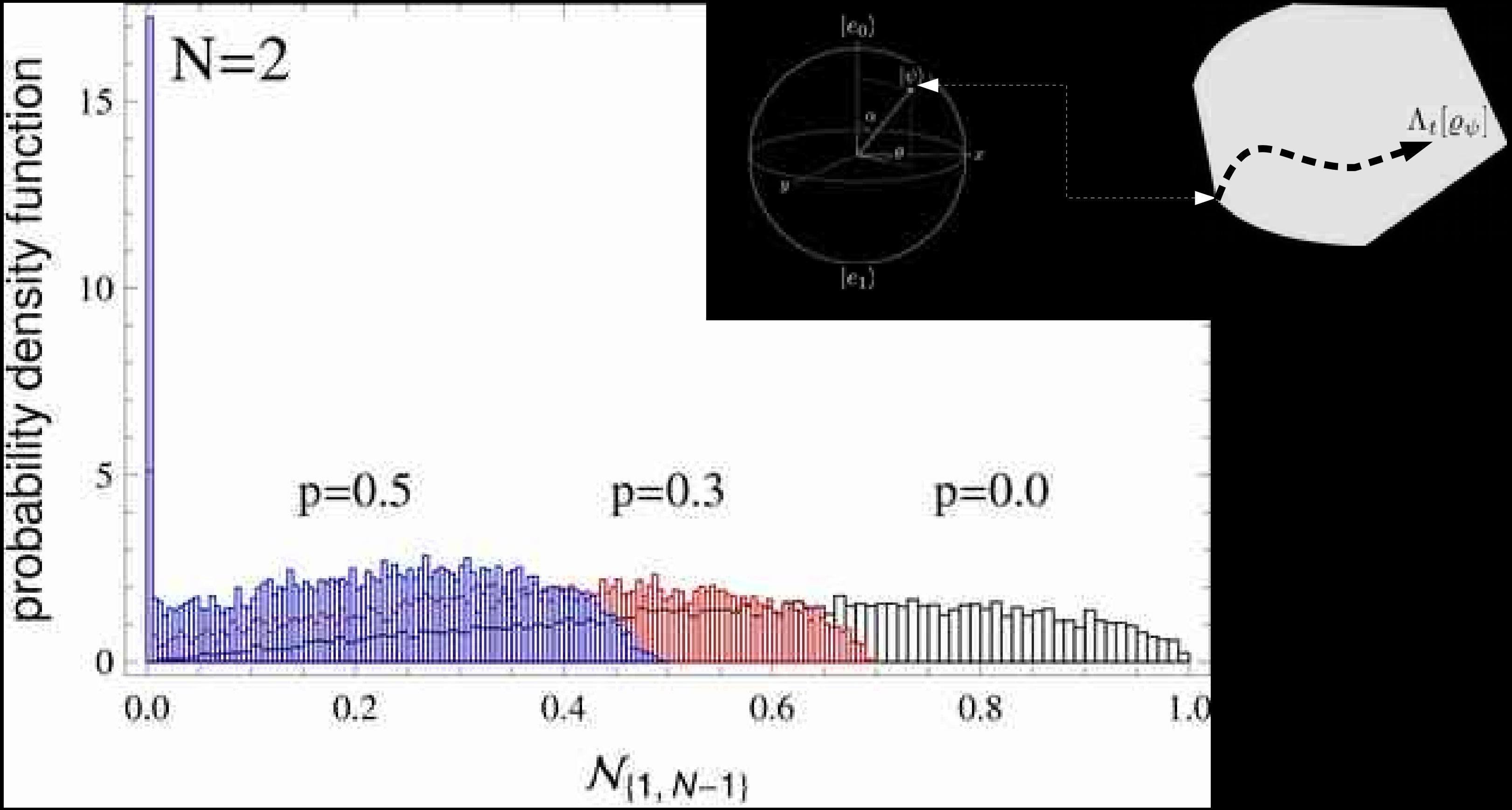
Consider a system in $\mathscr{H} = \mathscr{H}_A \otimes \cdots \otimes \mathscr{H}_N$.
If one uniformly samples pure states $|\psi\rangle \in \mathscr{H}$,
the chance that it is maximally entangled
approaches 1 exponentially fast with the total
dimension $d = d_A . d_B \ldots d_N$

P. Hayden, D. Leung, and A. Winter,
Comm. Math. Phys. 265, 95 (2006)

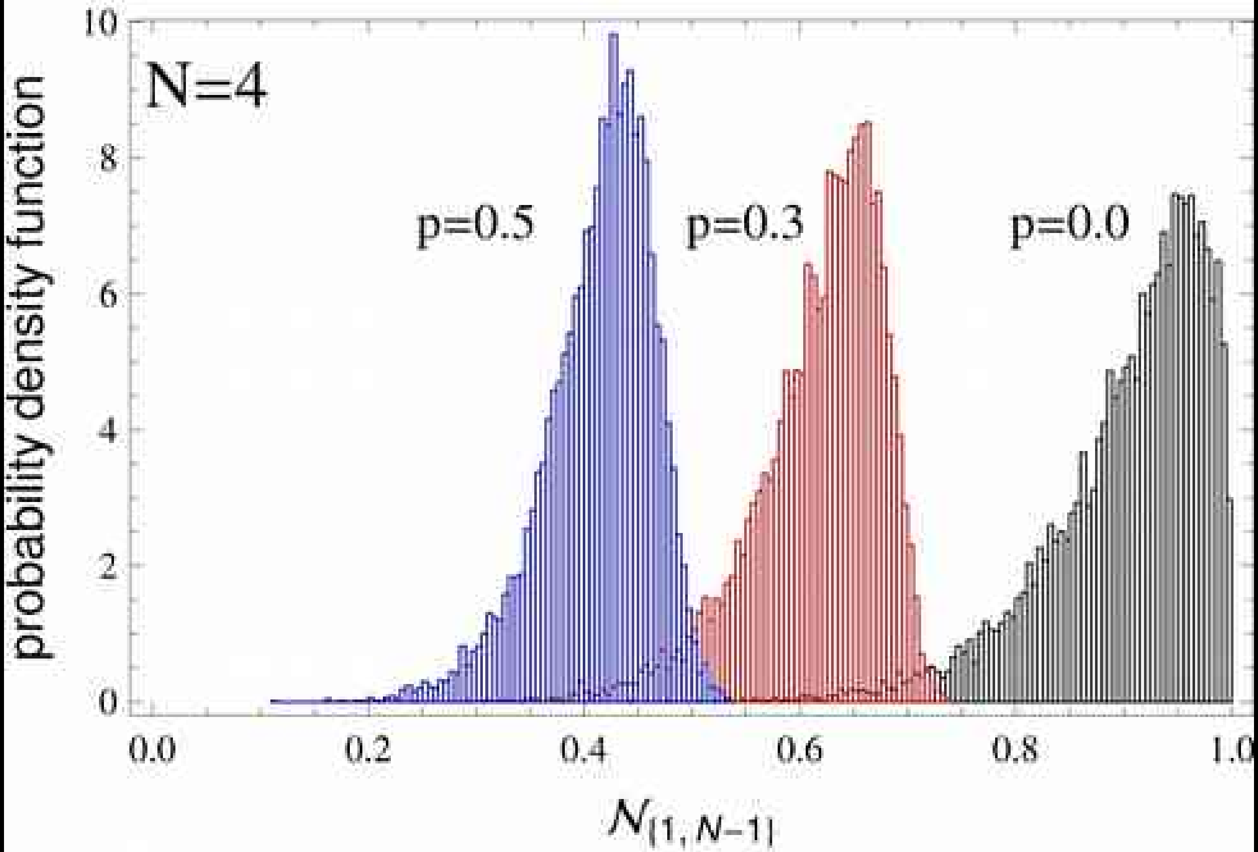


$$||\Lambda_t(\varrho) - \Lambda_t(\omega)||_1 \leq \eta_{\Lambda_t} \, ||\varrho - \omega||_1 \;\; ; \;\; \eta_{\Lambda_t} \leq 1$$

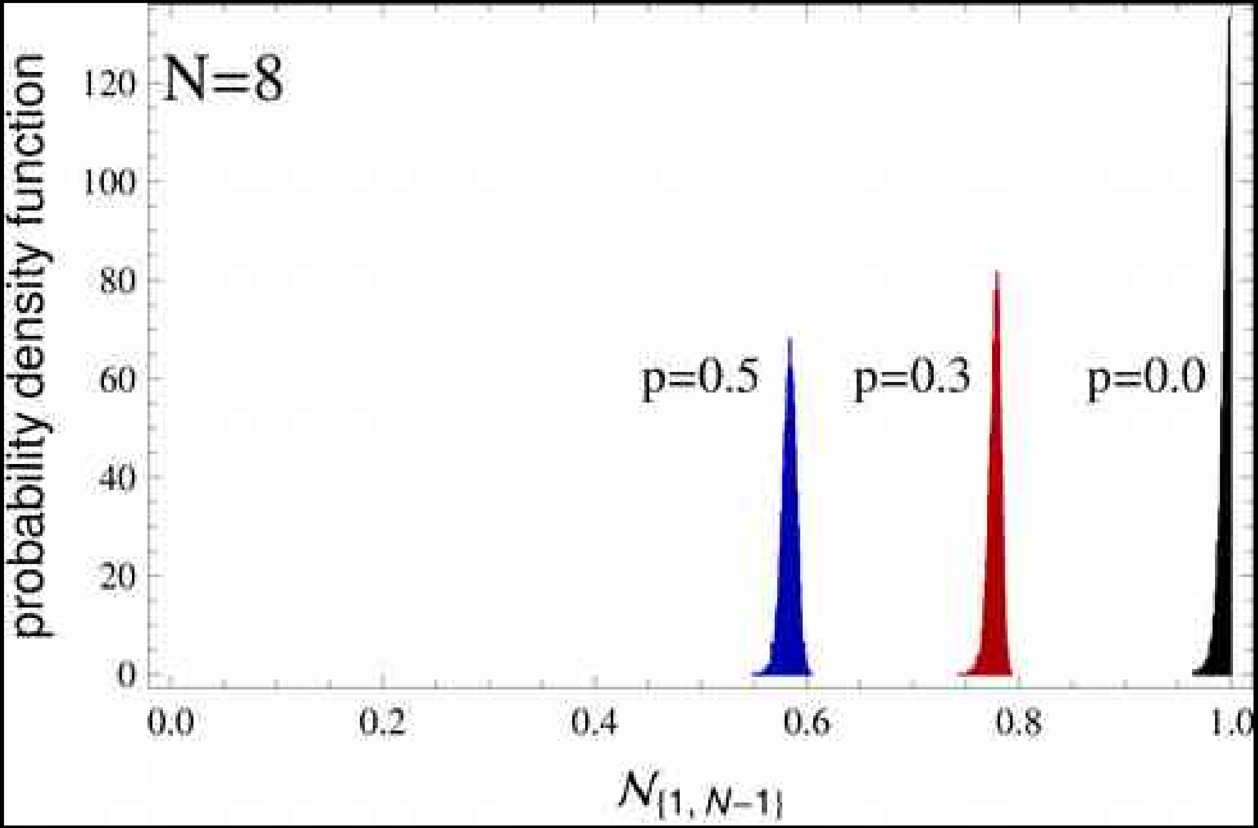
M. B. Ruskai, Rev. Math. Phys. 6, 1147 (1994).



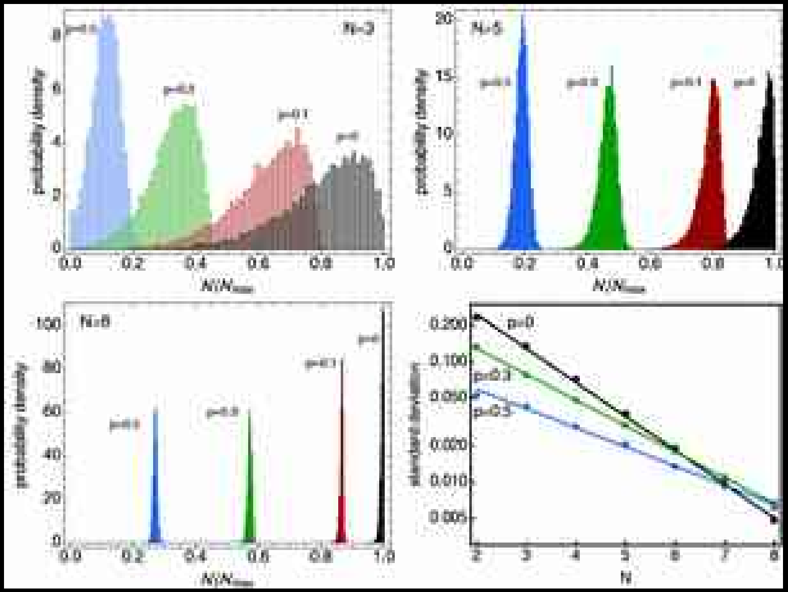
Dephasing $(p = 1 - e^{\Gamma t/2})$



Dephasing ($p = 1 - e^{\Gamma t/2}$)



Dephasing ($p = 1 - e^{\Gamma t/2}$)



$$\Pr\left(|E[\Lambda_t(|\psi\rangle\langle\psi|)] - E(t)| > \epsilon\right) \leq 4\exp\left(-C\frac{2d-1}{4\eta_E^2\eta_{\Lambda_t}^2}\epsilon^2\right)$$

with $C = (24\,\pi^2)^{-1}$ and $E(t) := \int d\chi\, E[\Lambda_t(|\chi\rangle\langle\chi|)]$

M. Tiersch, FdM, and A. Buchleitner, J. Phys. A 46, 085301 (2013)

Proof ingredients

- Lipschitz continuous entanglement measure

$$|E(\varrho) - E(\omega)| \leq \eta_E ||\varrho - \omega||_1$$

- Levy's lemma:

Let $f : \mathbb{S}^n \to \mathbb{R},$ with $|f(X) - f(Y)| \leq \eta ||X - Y|| \; \forall X, Y \in \mathbb{S}^n$

$$\Pr\left(|f(X) - \langle f \rangle| > \epsilon\right) \leq 4 \exp\left(-C \frac{n+1}{\eta^2} \epsilon^2\right)$$

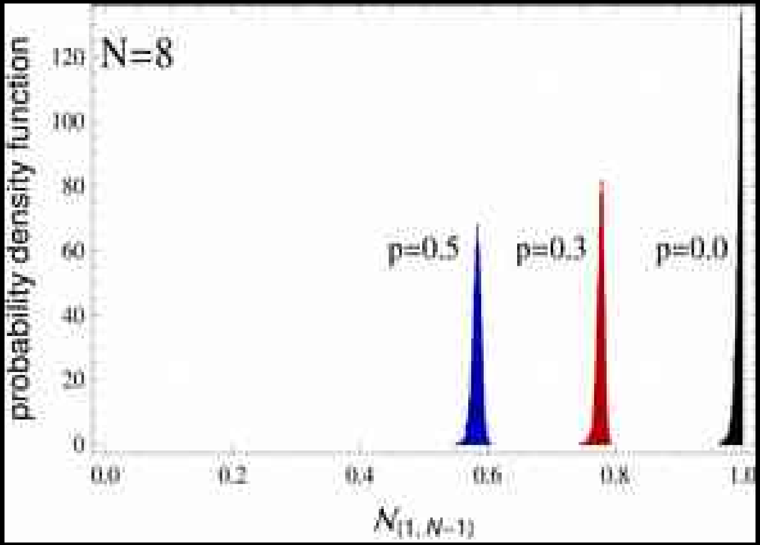
for a random $X \in \mathbb{S}^n,$ and $C > 0.$

Conclusions and open questions

Conclusions

- Entanglement theory is rich and interesting
- Determining entanglement is a hard task
- The complexity seems to be reduced when introducing dynamical aspects

Open questions



- Finding an equation of motion for the mean entanglement

- Use an efficient sampling distribution

Interested?


Open-system dynamics of entanglement: a key issues review

Leandro Aolita, Fernando de Melo and Luiz Davidovich

Abstract

1. Introduction

To appear in Rep. Prog. Phys.

78 pages
560 references
33 figures

Also available at
arXiv:1402.3713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concurrence
time
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evolution
information

Obrigado!