André Nachbin, IMPA

Ondas em Meios Desordenados

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Pesquisa em 3 frentes:

Parte A: Modelagem (FIS+MATE) e Análise Assintótica de EDPs/OPERADORES

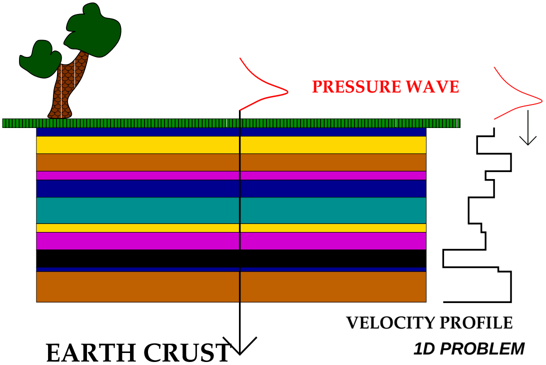
Parte B: Análise Assintótica de SOLUÇÕES de modelos REDUZIDOS

Parte C: Análise Numérica e Computação Científica

PRIMEIRA aplicação GEOFÍSICA com ONDAS em meios DESORDENADOS

DIFUSÃO APARENTE

MODELO ACÚSTICO 1D:

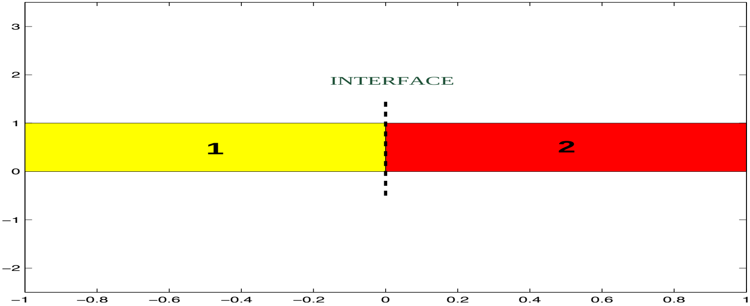


$$\frac{1}{\kappa(z/\varepsilon^2)}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial z}=0, \qquad \rho(z/\varepsilon^2)\frac{\partial u}{\partial t}+\frac{\partial p}{\partial z}=0,$$

VELO. ALEATÓRIA $c(z/\varepsilon^2)\equiv\sqrt{\kappa/\rho}$ **Dados:** $p(0,t)=u(0,t)=f(t/\varepsilon)$

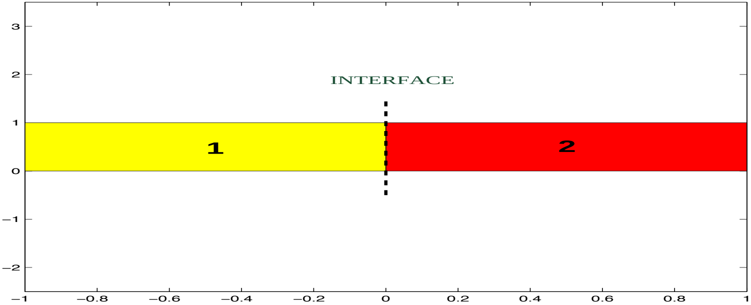
$1/\kappa\equiv$ **COMPRESSIBILIDADE da CROSTA terrestre** $\rho\equiv$ **DENSIDADE**

Situação com UMA discontinuidade:



$$
\begin{aligned}
1/\kappa(z)\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} &= 0 \\
\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} &= 0
\end{aligned}
$$

Impedância: $\zeta_i \equiv \sqrt{\rho_i \kappa_i}$; **e tempo de trânsito** $x = \int_0^z c^{-1}(s)ds$

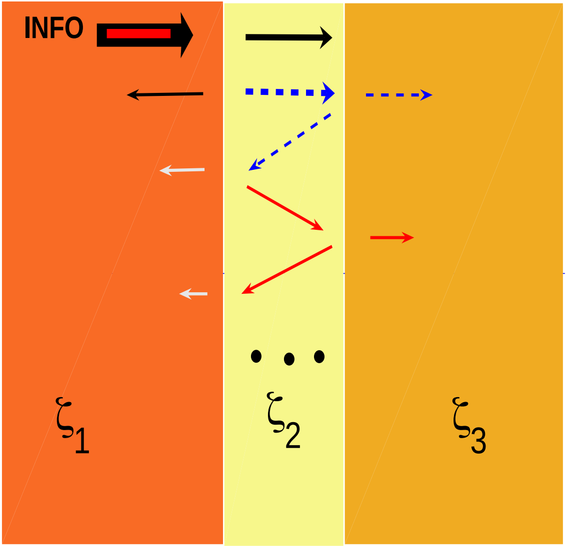


Continuidade de p **e** u**, e usando Invariantes de Riemann (const. ao longo de caracteríticas)...**

$$\textbf{TRANS.} \equiv \tau = \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2} \qquad \textbf{REFL.} \equiv \sigma = \frac{\zeta_2-\zeta_1}{\zeta_1+\zeta_2}$$

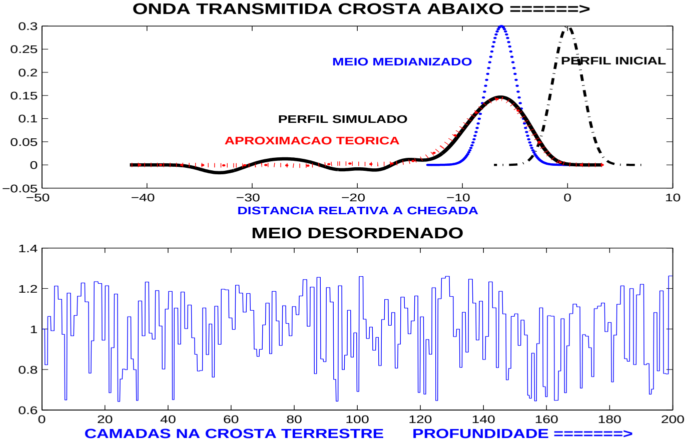
CONSERV: $\tau^2 + \sigma^2 = 1$

1D: VÁRIAS CAMADAS

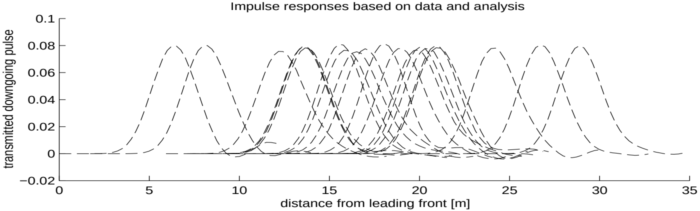


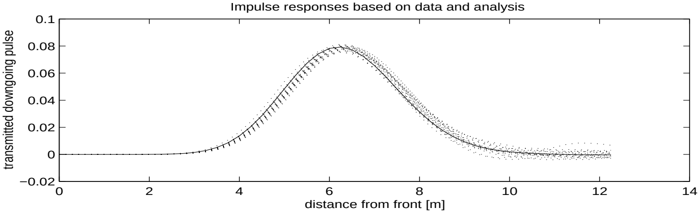
Ondas ACÚSTICAS ∼ Ondas AQUÁTICAS (”Shallow Water Theory”)

Temos EDPs + PROBABILIDADE ⇒ N. & Sølna, Phys. Fluids 2003



Resultado DETERMINÍSTICO a partir de modelagem ESTOCÁSTICA:

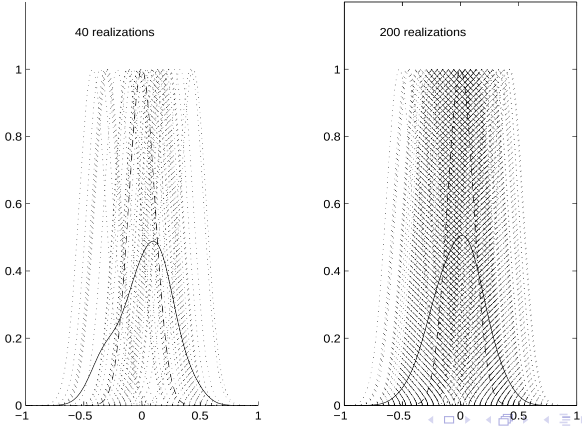




Teo. Central do Limite versus Teoria dos Campos Médios ('Wave Field'): Atenuação SUPER-estimada

EDP HIPERBÓLICA: advecção aleatória

pulso Gaussiano (dado inicial) c/ a velo. tendo uma distribuição normal.



EDOs ALEATÓRIAS com 1 ESCALA de TEMPO.

Teorema de Khasminskii(*): Sejam os PVIs $\omega \in (\Omega, \mathcal{A}, \mathcal{P})$

$$\frac{dx_\varepsilon}{dt} = \varepsilon F(t, x_\varepsilon; \omega), \qquad x_\varepsilon(0) = x_0$$

e

$$\frac{dy}{d\tau} = \overline{F}(y), \qquad y(0) = x_0,$$

onde $F(t, \cdot; \omega)$ **é um processo estocástico estacionário satisfazendo hipóteses de ergodicidade etc..., com**

$$\overline{F}(x) \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}\{F(t, x; \omega)\} dt.$$

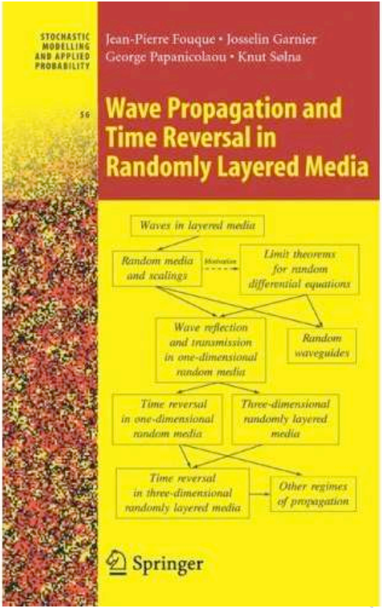
Então

$$\sup_{0 \le t} \mathbb{E}\{|x_\varepsilon(t) - y(t)|\} \sim \sqrt{\varepsilon} \quad \textbf{na escala de tempo } 1/\varepsilon.$$

(*) R.Z. Khasminskii, On stochastic processes defined by differential equations with a small parameter, Theory Prob. Applications, Volume XI (1966), pp.211-228.

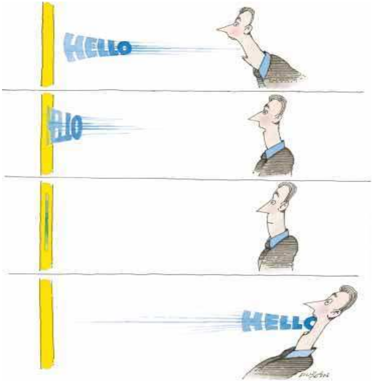
R.Z. Khasminskii, A limit-theorem for the solutions of differential equations with random right-hand sides, Theory Prob. Applications, Volume XI (1966), pp.390-406.

Lançado em meados de 2007



OUTRA aplicação GEOFÍSICA com ONDAS em meios DESORDENADOS

REFOCALIZAÇÃO via REVERSÃO TEMPORAL



DUSAN PETRICIC

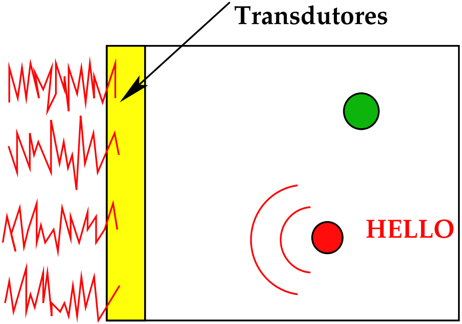
TIME-REVERSED ACOUSTICS

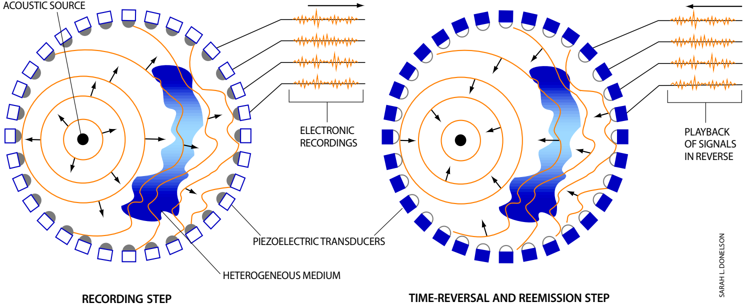
Arrays of transducers can re-create a sound and send it back to its source as if time had been reversed. The process can be used to destroy kidney stones, detect defects in materials and communicate with submarines

by Mathias Fink

In a room inside the Waves and Acoustics Laboratory in Paris is an array of microphones and loudspeakers. If you stand in front of this array and speak into it, anything you say comes back at you, but played in reverse. Your "hello" echoes—almost instantaneously—as "olleh." At first this may seem as ordinary as playing a tape backward, but the loudspeakers, the sound of the "olleh" converges onto your mouth, almost as if time itself had been reversed. Indeed, the process is known as time-reversed acoustics, and the array in front of you is acting as a "time-reversal mirror."

Such mirrors are more than just a novelty item. They have a range of applications, including destruction of tumors and


Transdutores
HELLO



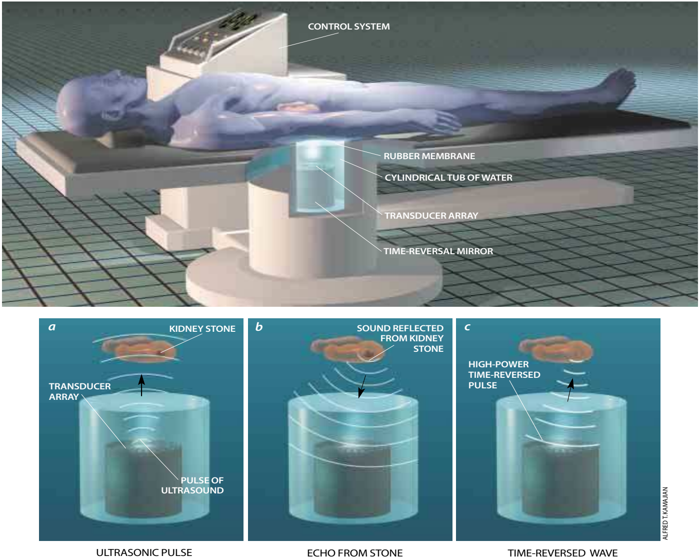
ACOUSTIC TIME-REVERSAL MIRROR operates in two steps. In the first step (*left*) a source emits sound waves (*orange*) that propagate out, perhaps being distorted by inhomogeneities in the medium. Each transducer in the mirror array detects the sound arriving at its location and feeds the signal to a computer. In the second step (*right*), each transducer plays back its sound signal in reverse in synchrony with the other transducers. The original wave is re-created, but traveling backward, retracing its passage back through the medium, untangling its distortions and refocusing on the original source point.

They can also be used for elegant experiments in pure physics.

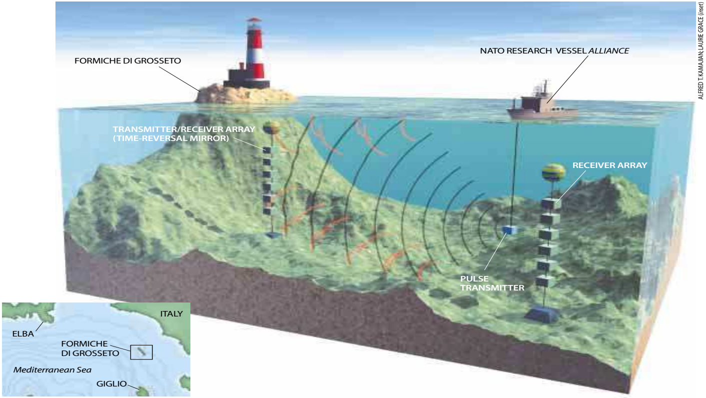
The magic of time-reversed acoustics is possible because sound is composed of waves. When you speak you produce vibrations in the air that travel like ripples on a pond spreading out from the point where a stone splashed in. A fundamental property of waves is that when two of them pass through the same location, they reinforce each other if

back on *exactly* the reversed trajectory, which again would totally alter the final outcome.

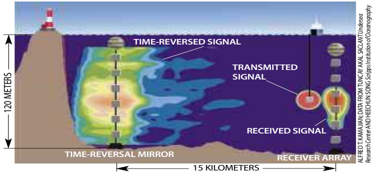
In contrast, wave propagation is linear. That is, a small change in the initial wave results in only a small change in the final wave. Likewise, reproducing the "final" wave, moving in reverse but with the inevitable small inaccuracies, will result in the wave propagating and re-creating the "ini-



KIDNEY STONES can be targeted and broken up with ultrasound by using the self-focusing property of a time-reversal mirror. An ultrasonic pulse emitted by one part of the array (*a*) produces a distorted echo from the stone (*b*). A powerful time-reverse of this echo passes through intervening tissues and organs, focuses back on the stone (*c*) and breaks it up. Iterating the procedure improves the focus and allows real-time tracking as the stone moves because of the patient's breathing.



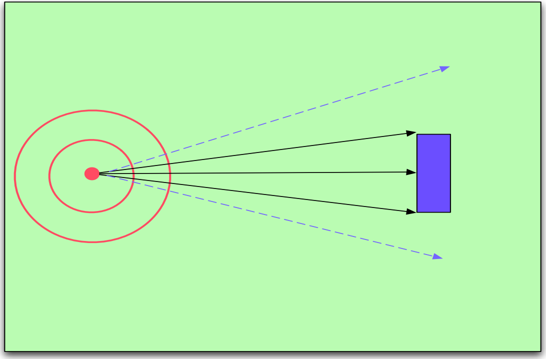
UNDERWATER COMMUNICATIONS can be enhanced by using time-reversed acoustics to focus a signal. This technique was demonstrated in water 120 meters deep near the island of Elba off the coast of Italy. A sound pulse was sent from the target location and recorded up to 30 kilometers away by an array of transponders, distorted by refraction and multiple reflections (*red*) from the surface and the seabed. The time-reversed signal sent by the array was well focused at the target location.



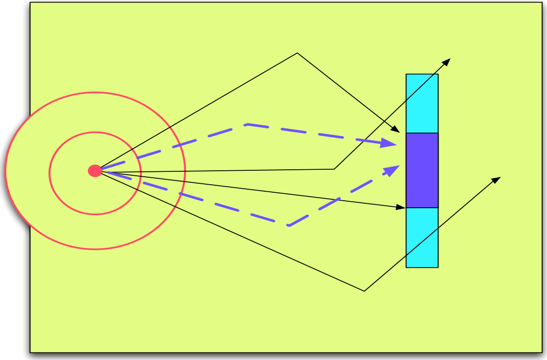
RESULTS from an underwater experimental run. Color contours indicate intensity of sound. The transmitted signal pulse (*red circle*) is greatly distorted at the time-reversal mirror, but when the time-reversed signal is played back (*at left*) it reproduces a focused pulse at the receiver array (*at right*).

cently researchers from the Scripps Institution of Oceanography in La Jolla, Calif., and the SACLANT Undersea Research Center in La Spezia, Italy, built and tested a 20-element TRM in the Mediterranean Sea off the coast of Italy [*see illustration above*]. Led by Tuncay Akal, William Hodgkiss and William A. Kuperman, they showed in water about 120 meters deep that their mirror could focus sound waves up to 30 kilometers away. In a result similar to the

ESQUEMATICAMENTE...



SUPER-RESOLUÇÃO!! “Multi-pathing”

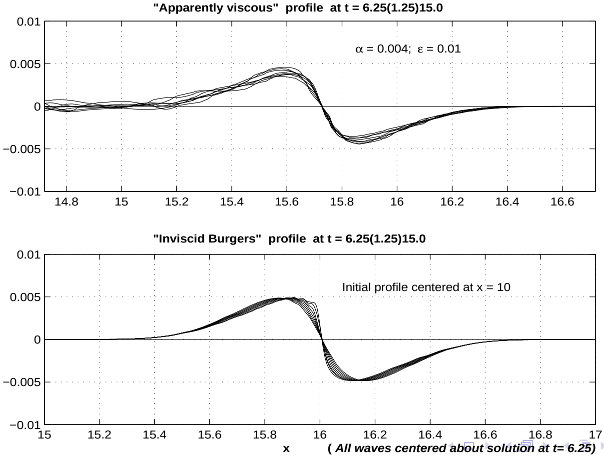


Time-reversal aperture enhancement, JP Fouque, K Solna - SIAM Multiscale Modeling and Simulation, 2003.
Super-resolution in time-reversal acoustics, P Blomgren, G Papanicolaou, H Zhao - The Journal of the Acoustical Society of America, 2002.

DESORDEM AJUDANDO!!

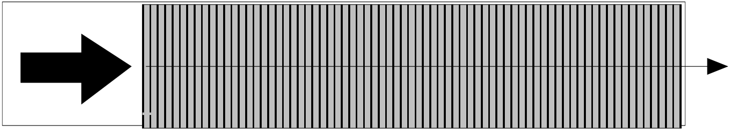
Forcante ALEATÓRIO $\Rightarrow$ **choque viscoso:** Fouque, Garnier & N., Physica D '04.

EDE ASSINTOTICAMENTE $\Rightarrow$ elevação da onda $\equiv \eta(x,t)$ governada por **Burgers' VISCOSA**



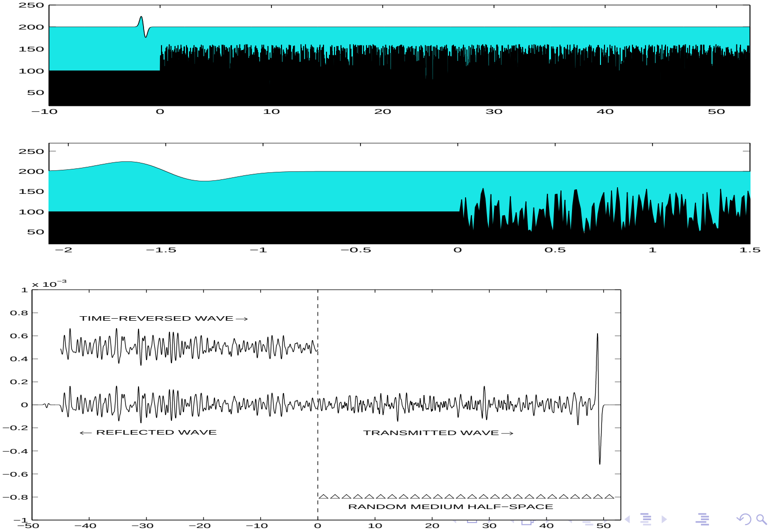
MEIO ALEATÓRIO muito LONGO:

estamos no regime de LOCALIZAÇÃO de Anderson

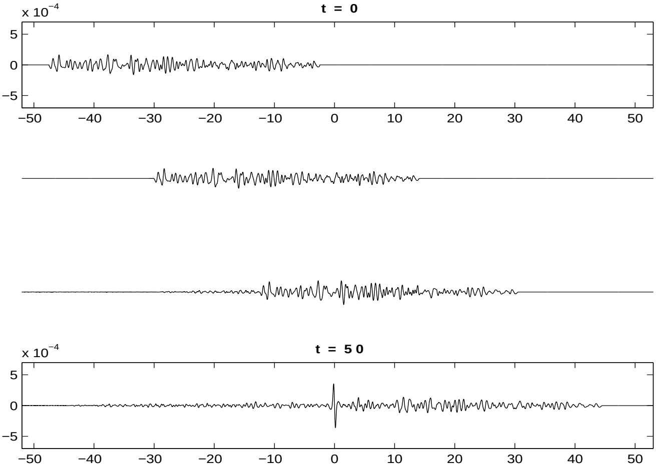


CENÁRIO para a TEORIA e SIMULAÇÕES: Reversão Temporal

Perfis típicos: Gaussianas, dGaussiana/dx e onda Solitária.

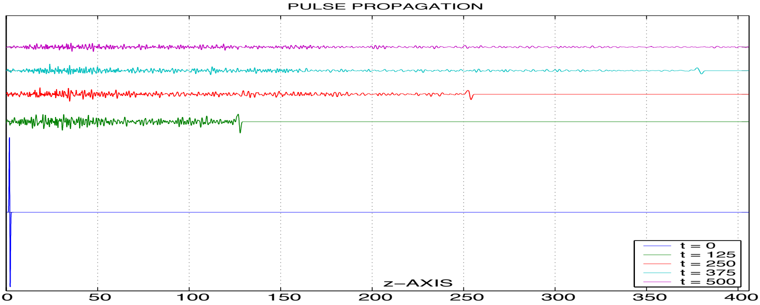


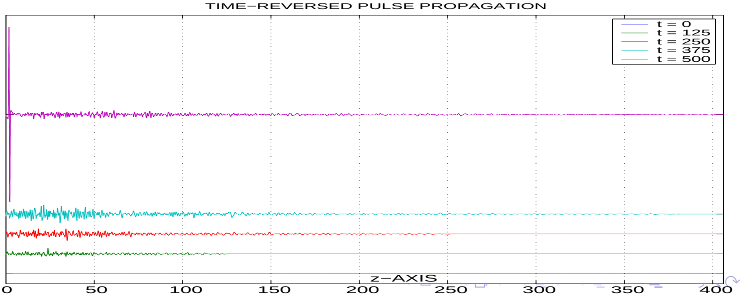
REFOCALIZAÇÃO 1D TSUNAMI



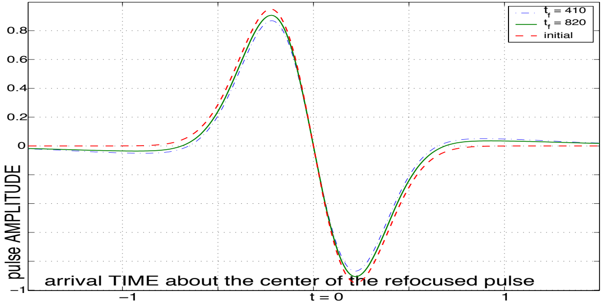
LOCALIZAÇÃO de ANDERSON:

Alfaro et al., Comm. Math. Sci., '07





Refocalização COMPLETA

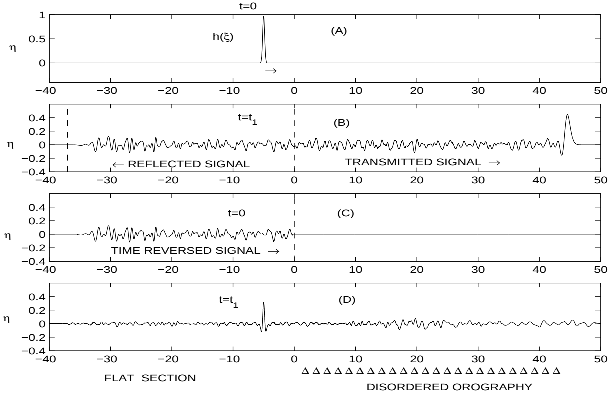


Regime Linear : Gaussiana

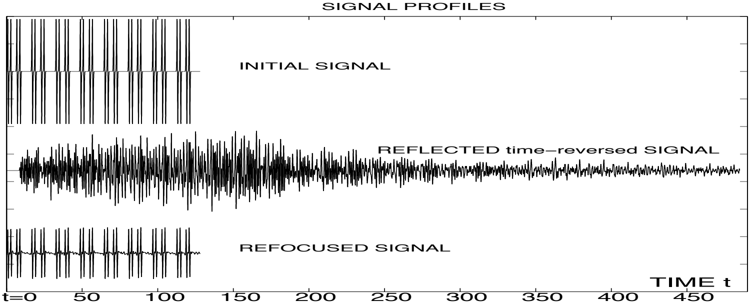
Clouet & Fouque, WMotion '97, Fouque & N. , SIAM MMS '04

$$\textbf{Pulso refocalizado} \equiv \eta^{TR}(t) = \frac{1}{2\pi}\int e^{-i\omega t}\overline{\hat{\eta}_0}(\omega)\left(\frac{\alpha_m\omega^2 t_0'}{1+\alpha_m\omega^2 t_0'}\right)\,d\omega.$$

$$\alpha_m = \int_0^\infty \mathbb{E}\{m(0)m(x)dx\} \quad M(s) = 1 + m(s)$$



"Embaralhando e desembaralhando" um bit stream



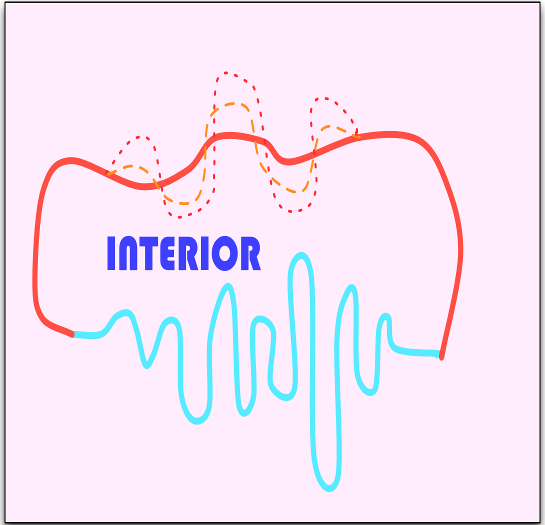
Parte A: .... da palestra.

MODELOS REDUZIDOS; PORQUE?

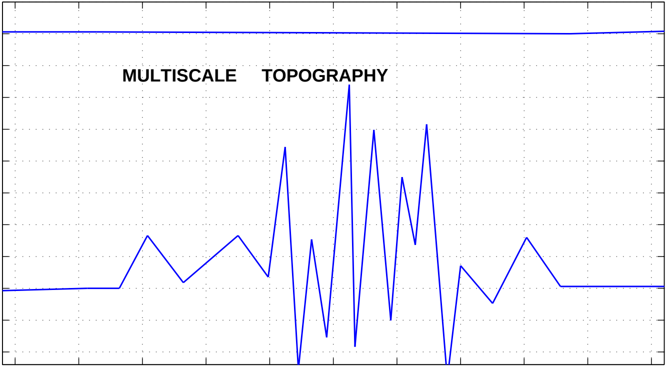
SIMULAÇÕES mais eficientes e ...
...melhor acesso à ANÁLISE/TEORIA MATEMÁTICA

Análise Assintótica de OPERADORES/EDPs:

MODELAGEM MATEMÁTICA



Típica geometria de uma topografia DESORDENADA com MÚLTIPLAS-ESCALAS:



Eq. de EULER ou Teoria do Potencial Não-Linear:

Equações em variáveis adimensionais:

$$\beta\phi_{xx} + \phi_{yy} = 0, \qquad \text{em} \qquad \Omega \equiv \text{CORPO FLUIDO},$$

com condições não-lineares na ...SUPERFÍCIE LIVRE

$$\begin{cases} \phi_t + \frac{\alpha}{2}(\phi_x^2 + \frac{1}{\beta}\phi_y^2) + \eta & = 0 \\ \eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_y & = 0 \end{cases} \qquad \text{em} \qquad y = \alpha\,\eta(x,t)$$

e uma cond. de Neumann na topografia DESORDENADA,

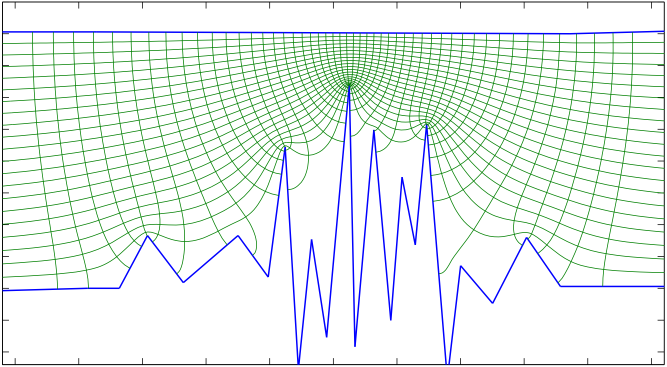
$$\frac{\beta}{\gamma}h'(\tfrac{x}{\gamma})\phi_x + \phi_y = 0 \qquad \text{ao longo de} \qquad y = -\sqrt{\beta}h(\tfrac{x}{\gamma}),$$

onde a TOPOGRAFIA DESORDENADA é dada através de h.

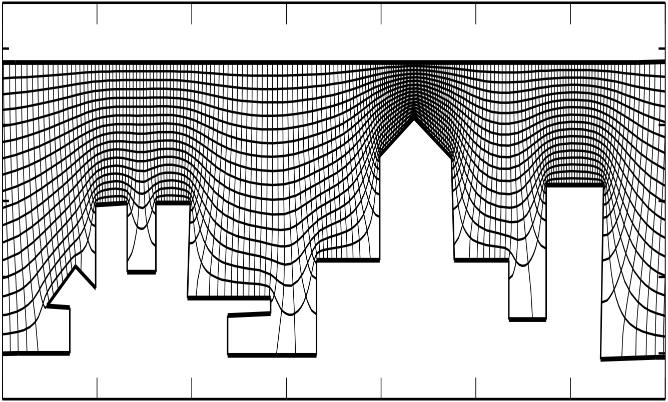
$\alpha \equiv (amplitude/profundd)$, $\beta \equiv (profundd/comprimento\ de\ onda)^2$, $\gamma \equiv (desordem/comprmt\ de\ onda)$

NÃO-LINEARIDADE DISPERSÃO DESORDEM

Típica geometria de uma topografia DESORDENADA com MÚLTIPLAS-ESCALAS:



Perfis Urbanos: problemas com Turbulência Urbana



COORDENADAS CURVILÍNEAS:

N. SIAP '03

$$\phi_{\xi\xi} + \phi_{\zeta\zeta} = 0, \qquad -\sqrt{\beta} < \zeta < S(\xi, t).$$

Na fronteira livre $\qquad \eta(x,t) \approx N(\xi(x,0),t)/M(\xi)$

$$N_t + \frac{\alpha}{|J|}\phi_\xi N_\xi - \frac{1}{|J|\sqrt{\beta}}\phi_\zeta = 0.$$

$$\phi_t + \frac{\alpha}{2|J|}(\phi_\xi^2 + \phi_\zeta^2) + \eta = 0.$$

Note que $\phi_\zeta = 0$ em $\zeta = -\sqrt{\beta}$.

$$(\partial_{\xi\xi} + \partial_{\zeta\zeta}) = |J|^2 \Delta_{xy} \Rightarrow |J| \equiv (y_\xi^2 + y_\zeta^2)_{|FS} \approx \quad y_\zeta^2(\xi, 0) + O(\varepsilon^2) \text{ (FRACA. N-LIN.)}$$

Na fronteira livre o coeficiente métrico é $M(\xi; \sqrt{\beta}, \gamma) \equiv y_\zeta(\xi, 0)$, onde

$$M(\xi; \sqrt{\beta}, \gamma) = \frac{\pi}{4\sqrt{\beta}} \int_{-\infty}^{\infty} \frac{h(x(\xi_o, -\sqrt{\beta})/\gamma)}{\cosh^2 \frac{\pi}{2\sqrt{\beta}}(\xi_o - \xi)} d\xi_o.$$

ANÁLISE ASSINTÓTICA de EDPs

Série de potências na viz. do fundo (transladado para) $\zeta = 0$ Whitham 1974

$$\phi(\xi,\zeta,t) = \sum_{n=0}^{\infty} \zeta^n \; f_n(\xi,t).$$

O potencial de velocidades (satisfiaz LAPLACE + NEUMANN)

$$\phi(\xi,\zeta,t) = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{(2n)!} \zeta^{2n} \frac{\partial^{2n} f(\xi,t)}{\partial \xi^{2n}} \approx \sum_{n=0}^{N} [\ldots]$$

Temos então $\phi(x,y,t) \equiv \cosh(k\sqrt{\beta}y)\exp(i(kx - \omega t))$

$$C^2(k) = \frac{\omega^2}{k^2} = \frac{1}{\sqrt{\beta}k} tanh(\sqrt{\beta}k)$$

(VELO. de FASE)^2 $$\approx 1 - \frac{1}{3}(\sqrt{\beta}k)^2 + \tfrac{2}{15}(\sqrt{\beta}k)^4 - \tfrac{17}{315}(\sqrt{\beta}k)^6 + O((\sqrt{\beta}k)^8)$$

Relação de dispersão truncada através da aprox. de Padé:
$C_a^2(k) = p(k)/q(k)$.

Madsen and Sørensen '92, Nwogu '93

Tomando a derivada de ϕ com respeito a ξ e avaliando a velo. em uma profundd. INTERMEDIÁRIA $\zeta = Z_0 \in [0,1]$

$$\phi_\xi(\xi, Z_0, t) \equiv u(\xi, t) = f_\xi - \frac{\beta}{2} {Z_0}^2 f_{\xi\xi\xi} + O(\beta^2)$$

CONDIÇÃO de FRONTEIRA LIVRE fica reduzida à família de equações BOUSSINESQ:

$$M(\xi)\eta_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi + \frac{\beta}{2}\left[\left(\mathbf{Z_0}^2 - \frac{1}{3}\right) u_{\xi\xi}\right]_\xi = 0$$

$$u_t + \eta_\xi + \alpha\left(\frac{u^2}{2M^2(\xi)}\right)_\xi + \frac{\beta}{2}(\mathbf{Z_0}^2 - 1)u_{\xi\xi t} = 0$$

família de equações BOUSSINESQ

$$M(\xi)\eta_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi + \frac{\beta}{2}\left[\left(\mathbf{Z_0}^2 - \frac{1}{3}\right) u_{\xi\xi}\right]_\xi = 0$$

$$u_t + \eta_\xi + \alpha\left(\frac{u^2}{2M^2(\xi)}\right)_\xi + \frac{\beta}{2}(\mathbf{Z_0}^2 - 1)u_{\xi\xi t} = 0$$

$$C^2 = \frac{\omega^2}{k^2} = \frac{1 - (\beta/2)(Z_0^2 - \frac{1}{3})k^2}{1 - (\beta/2)(Z_0^2 - 1)k^2}$$

$$\frac{\omega^2}{k^2} = \frac{1 + (\beta/15)k^2}{1 + 2(\beta/5)k^2}$$

...e para o valor especial $Z_0 = \sqrt{1/5}$

$$\approx 1 - \frac{1}{3}(\sqrt{\beta}k)^2 + \frac{2}{15}(\sqrt{\beta}k)^4 - \tfrac{4}{75}(\sqrt{\beta}k)^6 + O((\sqrt{\beta}k)^8).$$

Seja $Z_0 = \sqrt{2/3}$ e $u_\xi(\xi, t) = -M(\xi)\eta_t + O(\alpha, \beta)$:

$$(M(\xi)\eta)_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi - \frac{\beta}{6}(M(\xi)\eta)_{\xi\xi t} = 0$$

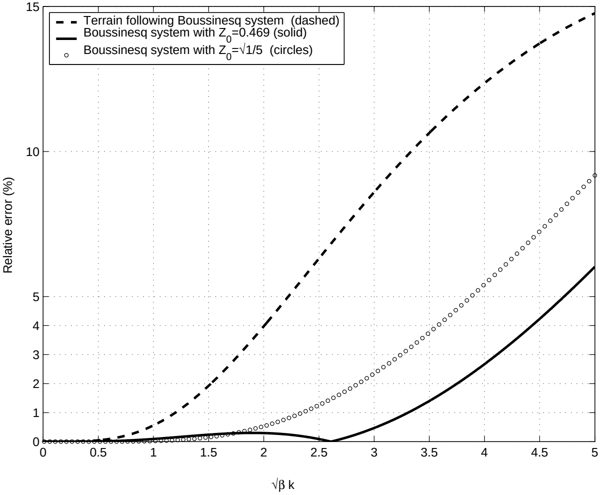
$$u_t + \eta_\xi + \alpha\left(\frac{u^2}{2M^2(\xi)}\right)_\xi - \frac{\beta}{6}u_{\xi\xi t} = 0$$

Quintero and Muñoz (Meth.Appl.Anal. '04) demonstraram existência, unicidade etc... após encontrarem uma integral de energia . Ferramentas semelhantes a Bona & Chen '98

$$\left(\mathtt{I} - \frac{\beta}{6}\partial_{\xi\xi}\right)^{-1}[U] = K_\beta * U, \qquad K_\beta(s) \equiv -\frac{1}{2}\sqrt{\frac{6}{\beta}}sign(s)e^{-\sqrt{6/\beta}|s|}$$

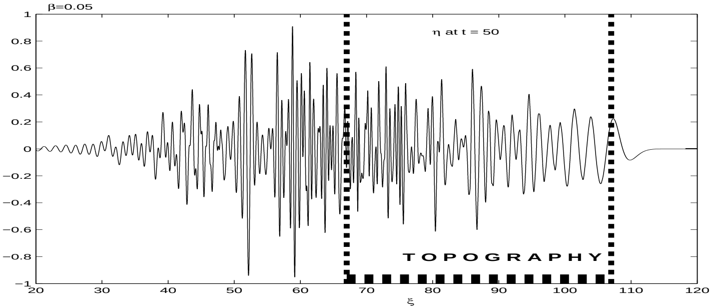
Mais pode ser feito! **Nwogu '93**

Outros valores: $Z_O = \sqrt{1/5} \approx 0.447$ e $Z_O = 0.469$: Muñoz & N., IMA '06



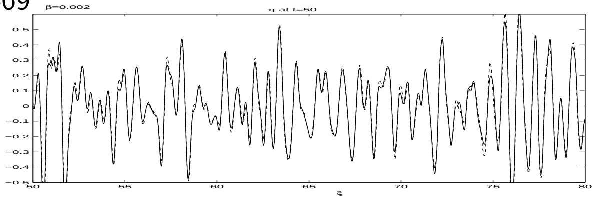
TOPOGRAFIA DESORDENADA

Comparamos modelos na JANELA ⇓

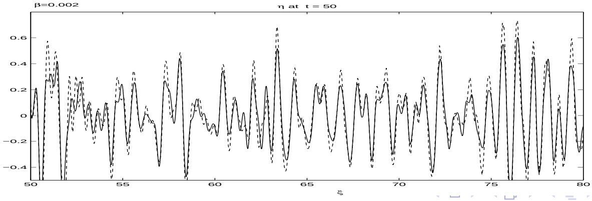


TOPOGRAFIA DESORDENADA: espalhamento múltiplo

$Z_O = 0.469$



$Z_O = \sqrt{2/3}$ melhor valor para Análise Funcional



NOVO $\Rightarrow$ MODELO ÓTIMO

...através de análise assintótica em múltiplas escalas.

Garnier, Kraenkel & N, PRE, October 2007

MODELO REDUZIDO (Boussinesq): $M \equiv K * h$

$$M\eta_t + \left[(1 + \frac{\alpha\eta}{M})u\right]_\xi - \frac{\beta}{2}(Z_0^2 - \frac{1}{3})\,[M\eta]_{\xi\xi t} = 0\,,$$

$$u_t + \eta_\xi + \alpha\left[\frac{u^2}{2M^2}\right]_\xi + \frac{\beta}{2}(Z_0^2 - 1)u_{\xi\xi t} = 0\,,$$

MODELO COMPLETO:

$\beta\phi_{xx} + \phi_{yy} = 0,$ em $\Omega \equiv$ CORPO FLUIDO periódico,

com condições não-lineares na ...FRONTEIRA LIVRE

$$\begin{cases} \phi_t + \frac{\alpha}{2}(\phi_x^2 + \frac{1}{\beta}\phi_y^2) + \eta & = 0 \\ \eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_y & = 0 \end{cases} \quad \text{em} \quad y = \alpha\,\eta(x,t)$$

e Neumann ao longo do fundo PERIÓDICO $h(x)$

$\frac{\beta}{\gamma}h'(\frac{x}{\gamma})\phi_x + \phi_y = 0$ ao longo de $y = -\sqrt{\beta}h(\frac{x}{\gamma}),$

Buscando uma KdV EFETIVA

Buscamos através de uma expansão multi-escala na forma

$$\eta(t,\xi) = \eta_0(t,\xi,\frac{\xi}{\varepsilon}) + \varepsilon\eta_1(t,\xi,\frac{\xi}{\varepsilon}) + \dots$$
$$u(t,\xi) = u_0(t,\xi,\frac{\xi}{\varepsilon}) + \varepsilon u_1(t,\xi,\frac{\xi}{\varepsilon}) + \dots$$

onde η_j e u_j são periódicas em $s = \xi/\varepsilon$ e as médias de η_1 e u_1 com respeito a s são zero.

MODELO BOUSSINESQ: A KdV EFETIVA é

$$h(x) = 1 + n(x) = 1 + n_1 \sin(kx)$$

$$\eta_{0_\tau} + \frac{3\alpha^*}{4}(\eta_0^2)_X + \frac{\beta^*}{6}\eta_{0XXX} = 0$$

onde $X = x - v^* t$ é um sistema de referência viajante. Em termos dominantes

$$\begin{aligned} v^{*2} &= 1 - \frac{n_1^2}{2}\frac{\sqrt{\beta_0}k}{\tanh(\sqrt{\beta_0}k)} \\ \alpha^* &= \alpha_0\left\{1 + \frac{n_1^2}{2}\left[\left(\frac{\sqrt{\beta_0}k}{\sinh(\sqrt{\beta_0}k)}\right)^2 + \frac{1}{2}\frac{\sqrt{\beta_0}k}{\tanh(\sqrt{\beta_0}k)}\right]\right\}, \\ \beta^* &= \beta_0\left\{1 + \frac{n_1^2}{2}\left[\left(\frac{3}{\beta_0 k^2} + 1 - \frac{3\beta_0 k^2}{4}({Z_0}^2 - \frac{1}{3})({Z_0}^2 - 1)\right)\left(\frac{\sqrt{\beta_0}k}{\sinh(\sqrt{\beta_0}k)}\right)^2 \right.\right. \\ &\qquad \left.\left. - \frac{5}{2}\frac{\sqrt{\beta_0}k}{\tanh(\sqrt{\beta_0}k)}\right]\right\}. \end{aligned}$$

são os PARÂMETROS da KdV EFETIVA .

TEORIA do POTENCIAL COMPLETA: A KdV EFETIVA é

Rosales & Papanicolaou '83

$$\eta_{0_\tau} + \frac{3\alpha^*}{4}(\eta_0^2)_X + \frac{\beta^*}{6}\eta_{0XXX} = 0$$

onde $X = x - v^* t$ **e**

$$\alpha^\star = \frac{1}{v^\star}\left(\alpha_0 + \frac{\alpha_0}{3}\left\langle A_x^2\right\rangle_s - \frac{2}{3v^\star}\left\langle n' D\right\rangle_b\right)$$

pode ser expandido, no caso SENOIDAL, dando lugar a

$$\alpha^\star = \alpha_0\left\{1 + \frac{n_1^2}{2}\left[\left(\frac{k\sqrt{\beta_0}}{\sinh(k\sqrt{\beta_0})}\right)^2 + \frac{1}{2}\frac{k\sqrt{\beta_0}}{\tanh(k\sqrt{\beta_0})}\right] + O(n_1^3)\right\}.$$

$$\beta^\star = \beta_0\left\{1 + \frac{n_1^2}{2}\left[3\frac{k\sqrt{\beta_0}}{\tanh^3(k\sqrt{\beta_0})} - \frac{11}{2}\frac{k\sqrt{\beta_0}}{\tanh(k\sqrt{\beta_0})}\right] + O(n_1^3)\right\}.$$

Casando as duas KdVs

...através dos respectivos coeficientes de dispersão $\beta^\star$.
Para $\beta_0 k^2$ pequenos, o casamento exato é obtido quando

$$Z_0 = \sqrt{\frac{2}{3} - \frac{1}{\sqrt{5}}}$$

Casando as duas KdVs

...através dos respectivos coeficientes de dispersão $\beta^{\star}$.
Para $\beta_0 k^2$ pequenos, o casamento exato é obtido quando

$$Z_0 = \sqrt{\frac{2}{3} - \frac{1}{\sqrt{5}}} \simeq 0.4685$$

Assim o modelo ótimo da família Boussinesq para a interação onda-microestrutura é

$$M\eta_t + \left[(1+\frac{\alpha\eta}{M})u\right]_\xi + \frac{\beta}{2}\left(\sqrt{\frac{1}{5}} - \frac{1}{3}\right)[M\eta]_{\xi\xi t} = 0\,,$$

$$u_t + \eta_\xi + \alpha\left[\frac{u^2}{2M^2}\right]_\xi - \frac{\beta}{2}\left(\sqrt{\frac{1}{5}} + \frac{1}{3}\right)u_{\xi\xi t} = 0\,.$$

Obrigado pela atenção.

IMPA, Rio de Janeiro.