Near-critical percolation and the geometry of diffusion fronts

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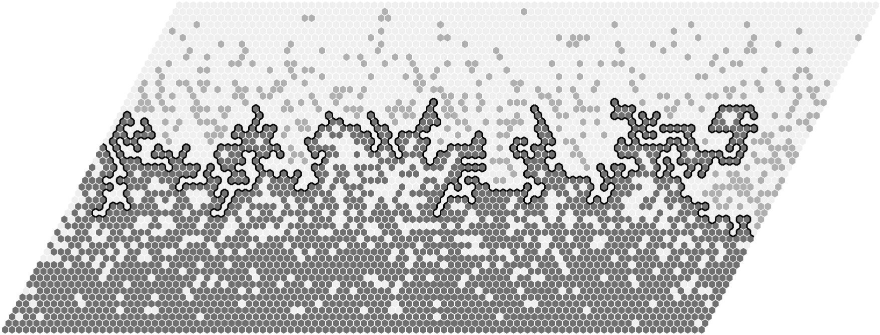
Motivation: geometry of diffusion fronts

The study of the geometry of **diffusion fronts** has been initiated by the physicists J.F. Gouyet, M. Rosso and B. Sapoval in 1985. They showed numerical evidence that such interfaces are fractal, and they measured the dimension $D_f = 1.76 \pm 0.02$.

To carry on simulations, they used the approximation that the status of the different sites (occupied / vacant) are **independent** of each other: they introduced an inhomogeneous percolation process with occupation parameter $p(z)$, where $p(z)$ is the probability of presence of a particle at site z (**Gradient percolation**).

Motivation: geometry of diffusion fronts



Motivation: geometry of diffusion fronts

One observes for this model various **critical exponents** (amplitude of the front, length...) that seem related to those of standard percolation.

We will explain how one can prove these observations, based on the recent works by G. Lawler, O. Schramm, W. Werner and S. Smirnov, that provide a very precise description of percolation near the critical point in 2 dimensions.

Motivation: geometry of diffusion fronts

Theoretical importance:

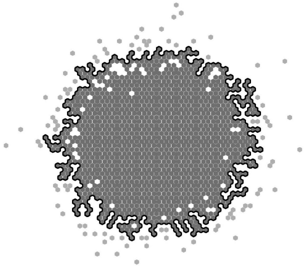
- **spontaneous** appearance of the percolation phase transition
- revealing of some critical exponents of percolation
- universality of the observed behavior (?)

Practical importance:

- efficient way of estimating p_c (B. Sapoval, B. Ziff. . .)

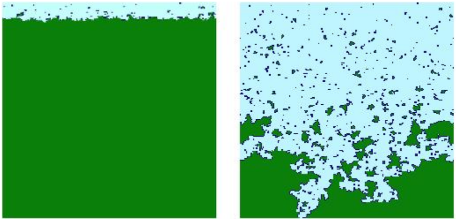
Motivation: geometry of diffusion fronts

We will then study a simple two-dimensional model where a large number of particles that start at a given site diffuse independently on a planar lattice. As the particles evolve, a **concentration gradient** appears, and the random interfaces that arise can be described by using our results for gradient percolation.



Motivation: geometry of diffusion fronts

Let us also mention a more "dynamical" model where random resistances are assigned to each site of a material (**Etching Gradient Percolation**), used for instance to explain the roughness of sea coasts (A. Gabrielli, A. Baldassarri, B. Sapoval).

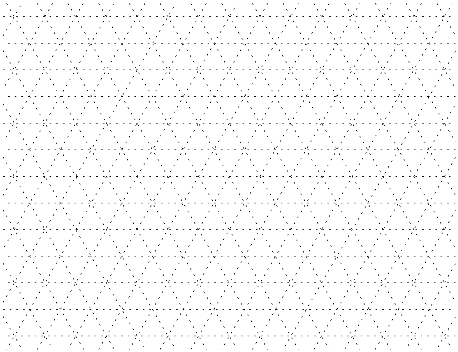


(Fig B. Sapoval)

Standard percolation background

Site percolation

We work in the plane, and we consider the *triangular lattice*:



Site percolation

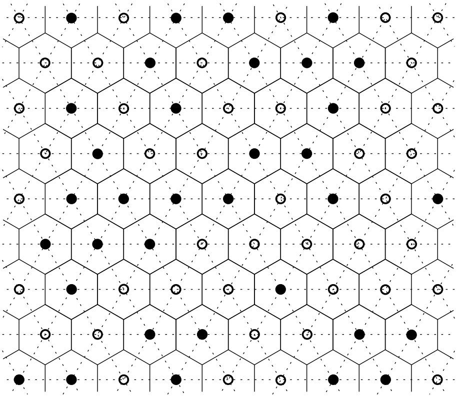
We fix a parameter $p \in [0,1]$, and we assume that:

- Each site is *occupied* (open / black) with probability p, *vacant* (closed / white) with probability $1-p$.
- The sites are *independent*.

The associated probability measure is denoted by $\mathbb{P}_p$.

Site percolation

We represent it as usual with hexagons:



Site percolation

We obtain this kind of pictures:

Remark

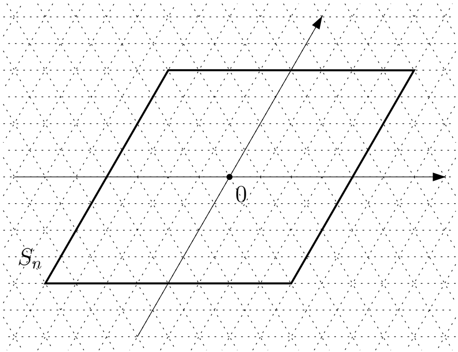
Why the triangular lattice?

We restrict to the triangular lattice, since at present, this is the only one for which the existence and the value of critical exponents have been proved.

However, the results presented here are likely to remain true on other lattices, like the square lattice $\mathbb{Z}^2$.

Notations

We use oblique coordinates:



Notations

We denote by

$$[a_1, a_2] \times [b_1, b_2]$$

the parallelogram of vertices $a_i + b_j e^{i\pi/3}$.

We use in particular

$$S_n = [-n, n] \times [-n, n]$$

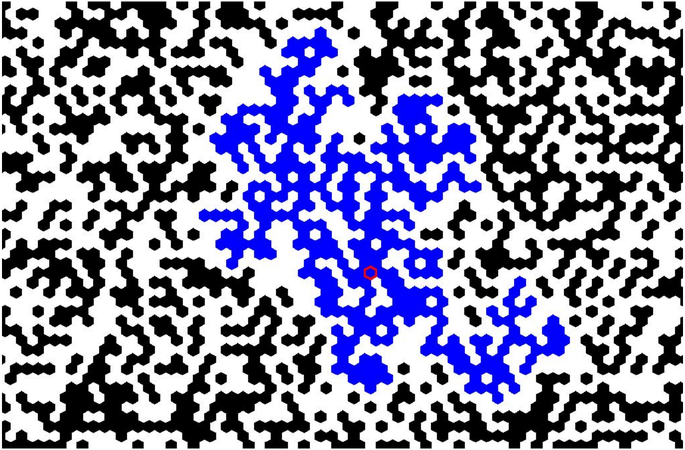
the "box of size n".

Cluster of a site

Two sites x et y are connected ($x \rightsquigarrow y$) if there exists a path from x to y composed only of black sites.

The set of sites connected to a site x is called the *cluster of* x. We will denote it by $C(x)$.

Cluster of a site



Existence of a phase transition at $p = 1/2$

Percolation features a *phase transition*, at $p = 1/2$ on the triangular lattice:

- If $p < 1/2$: a.s. no infinite cluster (*sub-critical* regime).
- If $p > 1/2$: a.s. a *unique* infinite cluster (*super-critical* regime).

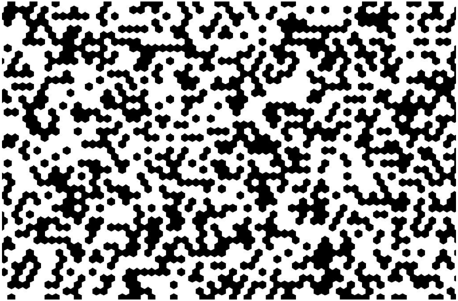
If $p = 1/2$: *critical* regime, a.s. no infinite cluster.

Exponential decay

In *sub-critical* regime ($p < 1/2$), there exists a constant $C(p)$ such that

$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n) \leq e^{-C(p)n}.$$

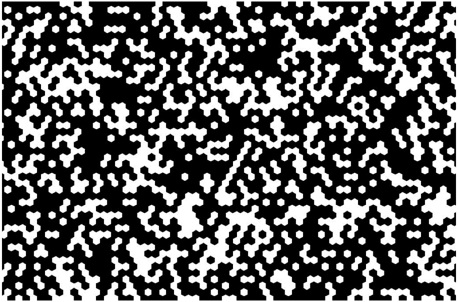
$\Rightarrow$ Fast "decorrelation" of distant points (speed *depends on p* !).



Exponential decay

In *super-critical* regime ($p > 1/2$), we have similarly

$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n | 0 \not\rightsquigarrow \infty) \leq e^{-C(p)n}.$$



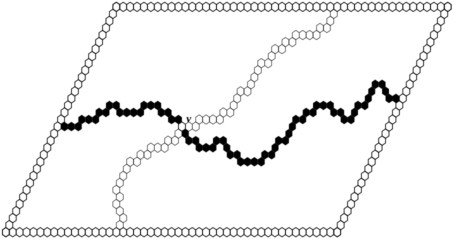
Critical regime

At the critical point $p = 1/2$, "there is no characteristic length": when we take some distance (scaling), we still observe the same behavior.

Critical regime

For symmetry reasons, we have for example:

$$\mathbb{P}_{1/2}(\text{crossing } [0, n] \times [0, n] \text{ from left to right}) = 1/2.$$



Critical regime

This implies the Russo-Seymour-Welsh theorem, which is a key tool for studying critical percolation:

Theorem (Russo-Seymour-Welsh)

For each $k \geq 1$, there exists $\delta_k > 0$ such that

$$\mathbb{P}_{1/2}(\textit{crossing } [0, kn] \times [0, n] \textit{ from left to right}) \geq \delta_k.$$

Near-critical percolation

Two main ingredients:

(1) Study of critical percolation

(2) Scaling techniques

$\Rightarrow$ Description of percolation near the critical point.

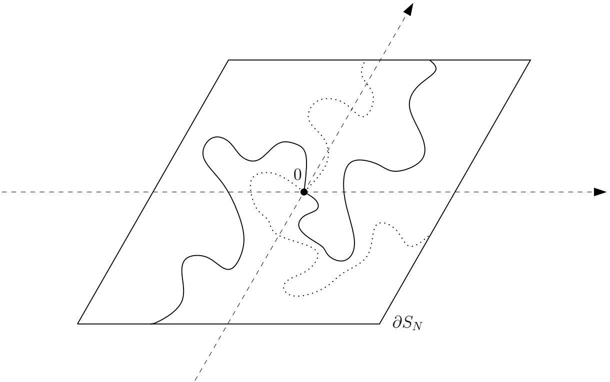
1st ingredient: study of critical percolation

A precise description of critical percolation was made possible by the introduction of SLE processes in 1999 by O. Schramm, and its subsequent study by G. Lawler, O. Schramm et W. Werner.

Another important step: conformal invariance of critical percolation in the scaling limit (S. Smirnov - 2001), that allows to go from discrete to continuum.

Arm events

We use in particular the “arm-events”:



Arm events

Their probabilities decay like a power law, described by the "j-arm exponents":

Theorem (Lawler, Schramm, Werner, Smirnov)

We have

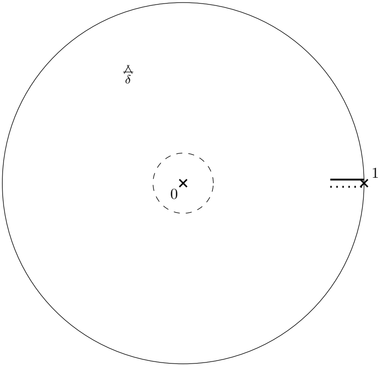
$$\mathbb{P}_{1/2}(0 \rightsquigarrow \partial S_n) \approx n^{-5/48}$$

and for each $j \geq 2$, for every fixed (non constant) sequence of "colors" for the j arms,

$$\mathbb{P}_{1/2}(j \text{ arms } \partial S_j \rightsquigarrow \partial S_n) \approx n^{-(j^2-1)/12}.$$

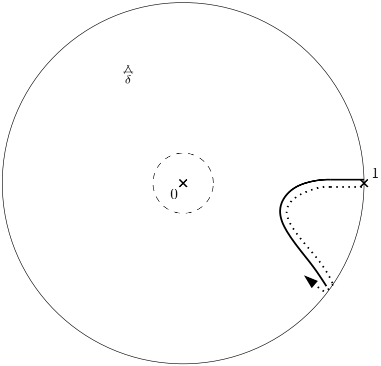
Study of critical percolation

To describe the discrete process, we try to understand its scaling limit. We define a radial exploration process “dynamically”:



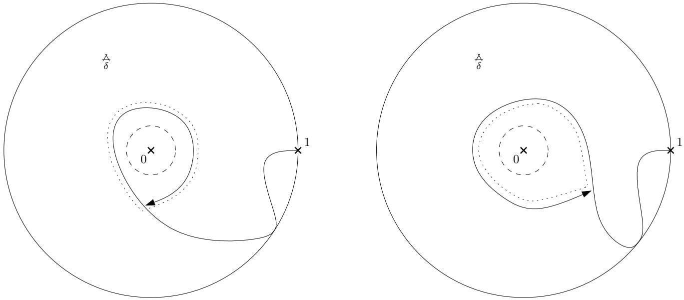
Study of critical percolation

We do not fix a priori boundary conditions, we choose them so that the process "bounces":



Study of critical percolation

It sometimes closes a clockwise loop or an anti-clockwise loop:



In this case we start again in the new domain (with no boundary condition).

Study of critical percolation

We have the following connection:

Theorem (Smirnov, Camia-Newman)

When the mesh $\delta \to 0$, the radial exploration process converges to a radial SLE_6.

$\Longrightarrow$ Link between arm events and some events for radial SLE_6 (ex: reaching the inner boundary of the annulus without closing any loop).

2nd ingredient: scaling techniques (H. Kesten)

For any $\epsilon \in (0, 1/2)$ fixed, we define a **characteristic length** ($\mathcal{C}_H$ denotes existence of a left-right crossing):

Definition

$$L_\epsilon(p) = \begin{cases} \min\{n \;\; s.t. \;\; \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq \epsilon\} & \textit{if } p < 1/2 \\ \min\{n \;\; s.t. \;\; \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \geq 1 - \epsilon\} & \textit{if } p > 1/2 \end{cases}$$

"It still looks close to critical percolation"

For instance, the RSW theorem remains true for parallelograms of size $\leq L_\epsilon(p)$, and the probability to observe a path $0 \rightsquigarrow \partial S_n$ remains of the same order of magnitude.

We will need:

Lemma

For any $j \geq 1$ and any fixed colors,

$$\mathbb{P}_p(j \text{ arms } \partial S_j \rightsquigarrow \partial S_n) \asymp \mathbb{P}_{1/2}(j \text{ arms } \partial S_j \rightsquigarrow \partial S_n)$$

uniformly in p, $n \leq L_\epsilon(p)$.

"We quickly become sub-critical"

We have the following lemma, showing exponential decay with respect to $L_\epsilon(p)$ (control of speed for variable p):

Lemma

There exist constants $C_1, C_2 > 0$ such that for each n, each $p < 1/2$,

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq C_1 e^{-C_2 n / L_\epsilon(p)}.$$

This lemma implies in particular if $p > 1/2$:

$$\mathbb{P}_p[0 \rightsquigarrow \infty] \asymp \mathbb{P}_p[0 \rightsquigarrow \partial S_{L_\epsilon(p)}]$$

At this distance, we are already "significantly" far from the origin.

Scaling techniques

To sum up, $L_\epsilon(p)$ is at the same time:

- a scale on which everything looks like critical percolation.
- a scale at which connectivity properties start to change drastically.

We can also prove that

$$L_\epsilon(p) \asymp L_{\epsilon'}(p)$$

for any $\epsilon, \epsilon' \in (0, 1/2)$.

Consequences for the characteristic functions

These ingredients allow to obtain the critical exponents of standard percolation, associated to the characteristic functions used to describe macroscopically the model (ξ, χ, θ ...). By counting pivotal sites,

$$|p - 1/2|(L_\epsilon(p))^2 \mathbb{P}_{1/2}(0 \rightsquigarrow^4 \partial S_{L_\epsilon(p)}) \asymp 1.$$

Hence,

$$L_\epsilon(p) \approx |p - 1/2|^{-4/3} \qquad (p \to 1/2).$$

The density $\theta(p)$ of the infinite cluster satisfies $(5/36 = (-5/48) \times (-4/3))$:

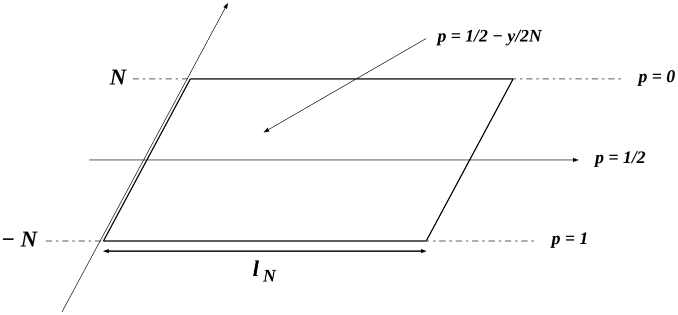
$$\theta(p) \approx (p - 1/2)^{5/36} \qquad (p \to 1/2^+).$$

Gradient Percolation

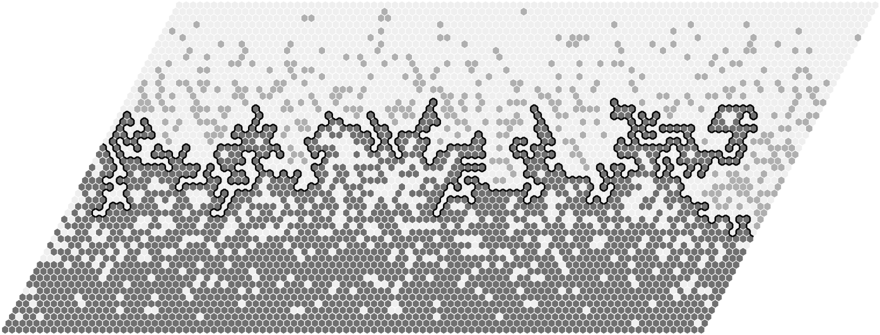
Gradient percolation: setting

We consider a strip $\mathcal{S}_N = [0, \ell_N] \times [-N, N]$, of finite width $2N$, in which the percolation parameter decreases linearly in y:

$$p(z) = 1/2 - y/2N.$$



Gradient percolation: setting



Gradient percolation: setting

With this convention, all the sites on the lower boundary are occupied ($p = 1$), all the sites on the upper boundary are vacant ($p = 0$).

$\Rightarrow$ Two different regions appear:

- At the bottom of $\mathcal{S}_N$, the parameter is close to 1, we are in a super-critical region and most occupied sites are connected to the bottom: "big" cluster of occupied sites.
- At the top of $\mathcal{S}_N$, the parameter is close to 0, we are in a sub-critical region and most vacant sites are connected to the top (by vacant paths): "big" cluster of vacant sites.

Gradient percolation: setting

The characteristic phenomenon of this model is the appearance of a unique “front”, an interface touching simultaneously these two clusters.

Hypothesis on ℓ_N: we will assume that for two constants $\epsilon, \gamma > 0$,

$$N^{4/7+\epsilon} \leq \ell_N \leq N^{\gamma}$$

Thus $\ell_N = N$ is OK.

Heuristics

The critical behavior of this model remains localized in a “critical strip” around $p = 1/2$, a strip in which we can consider percolation as almost critical.

We get away from the critical line $p = 1/2$: the characteristic length associated to the percolation parameter decreases $\Longrightarrow$ at some point, it gets of the same order as the distance from the critical line. This distance is the width σ_N of the critical strip

$$\sigma_N = L_\epsilon(1/2 \pm \sigma_N/2N).$$

The vertical fluctuations of the front are of order σ_N.

The exponent for $L_\epsilon(p)$ implies that $\sigma_N \approx N^{4/7}$.

Heuristics

Hence, we expect:

- $\longrightarrow$ **uniqueness** of the front.
- $\longrightarrow$ **decorrelation** of points at horizontal distance $\gg \sigma_N$.
- $\longrightarrow$ **width** of the front of the order of σ_N.

Uniqueness

There exists with probability very close to 1 a unique front, that we denote by $\mathcal{F}_N$:

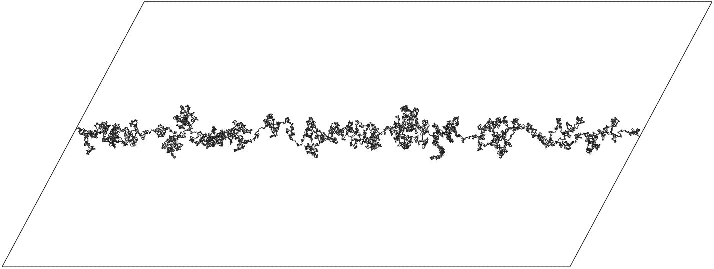
Lemma (N.)

There exists $\delta' > 0$ such that for each N sufficiently large,

$$\mathbb{P}(\textit{the boundaries of the two "big" clusters coincide}) \geq 1 - e^{-N^{\delta'}}.$$

Consequence: a site x is on the front *iff* there exist two arms, one occupied to the bottom of $\mathcal{S}_N$, and one vacant to the top. Moreover, this is a **local** property (depending on a neighborhood of x of size $\approx \sigma_N$) $\Rightarrow$ **decorrelation** of points at horizontal distance $\gg \sigma_N$.

Width of the front



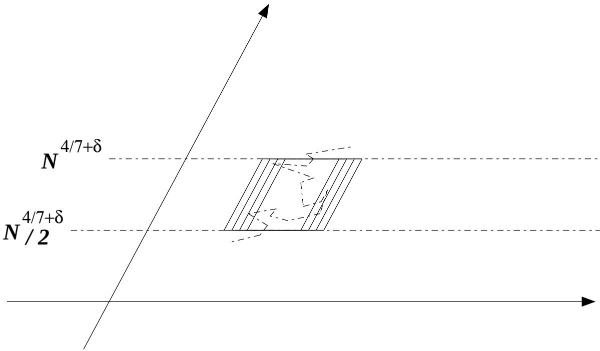
Width of the front

The scale $\sigma_N \approx N^{4/7}$ is actually the order of magnitude of the vertical fluctuations:

Theorem (N.)

- *For each* $\delta > 0$*, there exists* $\delta' > 0$ *such that for* N *sufficiently large,*
$$\mathbb{P}(\mathcal{F}_N \subseteq [\pm N^{4/7-\delta}]) \leq e^{-N^{\delta'}}.$$
- *For each* $\delta > 0$*, there exists* $\delta' > 0$ *such that for* N *sufficiently large,*
$$\mathbb{P}(\mathcal{F}_N \not\subseteq [\pm N^{4/7+\delta}]) \leq e^{-N^{\delta'}}.$$

Width of the front



Representative example: length of the front

To estimate a quantity related to the front:

(1) Only the edges in the critical strip must be counted.

(2) For these edges, being on the front is equivalent to the existence of two arms of length σ_N:

$$\mathbb{P}(e \in \mathcal{F}_n) \approx (\sigma_N)^{-1/4} \approx N^{-1/7}$$

(2 arm exponent: 1/4).

Representative example: length of the front

Thus, for the length T_N of the front:

Proposition (N.)

For each $\delta > 0$, we have for N sufficiently large:

$$N^{3/7-\delta}\ell_N \leq \mathbb{E}[T_N] \leq N^{3/7+\delta}\ell_N.$$

For $\ell_N = N$, this gives $\mathbb{E}[T_N] \approx N^{10/7}$.

Noteworthy property: In a box of size σ_N, approximately N points are located on the front.

Variance of T_N

The decorrelation of points at horizontal distance $\gg N^{4/7}$ implies:

Theorem (N.)

If for some $\epsilon > 0$, $\ell_N \geq N^{4/7+\epsilon}$, then

$$\frac{T_N}{\mathbb{E}[T_N]} \longrightarrow 1 \quad \text{in } L^2, \text{ as } N \to \infty.$$

$\Rightarrow$ Concentration of T_N around $\mathbb{E}[T_N] \approx N^{3/7}\ell_N$.

Outer boundaries of the front

We can introduce the lower and upper boundaries of the front: we denote by U_N^+ and U_N^- their respective lengths. The proof of the results on the length T_N can easily be adapted, and the 3-arm exponent (equal to $2/3$) gives:

$$U_N^{\pm} \approx N^{4/21} \ell_N .$$

Estimating p_c

We introduce the mean height:

$$Y_N = \frac{1}{T_N} \sum_e y_e \mathbb{I}_{e \in \mathcal{F}_N},$$

and we normalize it:

$$\tilde{Y}_N = \frac{1}{2} + \frac{Y_N}{2N}.$$

For symmetry reasons, $\mathbb{E}[\tilde{Y}_N] = 1/2$, and the decorrelation property implies that:

$$\mathrm{Var}(\tilde{Y}_N) \leq \frac{1}{N^{2/7-\delta} \ell_N}.$$

Estimating p_c

But on other lattices, like $\mathbb{Z}^2$?
We still have $L_\epsilon(p) \leq |p - p_c|^{-A}$. $\Rightarrow$ The front still converges toward p_c.

The results presented here come from the exponents of standard percolation.
$\Rightarrow$ For *universality* reasons, we can think that the critical exponents remain the same on other lattices, like the square lattice.

Estimating p_c

Question: Behavior of $\tilde{Y}_N$ when we lose symmetry ? We probably still have (decorrelation) if ℓ_N is sufficiently large ($\ell_N = N^2$ for example):

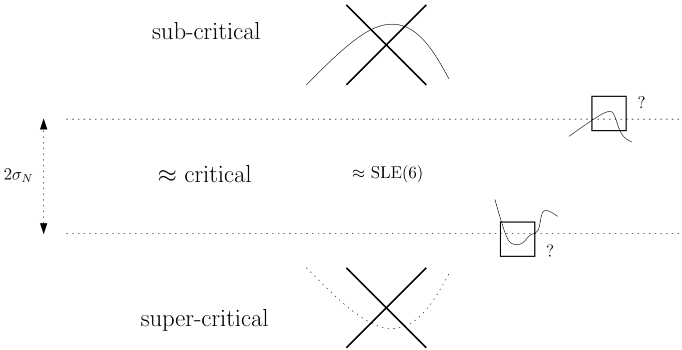
$$\tilde{Y}_N \approx \mathbb{E}[\tilde{Y}_N],$$

and also (localization):

$$p_c - N^{-3/7} \leq \mathbb{E}[\tilde{Y}_N] \leq p_c + N^{-3/7}.$$

But we can hope for a much better bound (in $1/N$ for instance).

Summary



Existence of scaling limits

Using standard arguments due to M. Aizenman and A. Burchard, one can show the existence of *scaling limits*. The right way to scale is by using the characteristic length

$$\sigma_N^\epsilon = \sup\{\sigma \quad \text{s.t.} \quad L_\epsilon(1/2 \pm \sigma/2N) \geq \sigma\}.$$

One can check that

$$\sigma_N^\epsilon \asymp \sigma_N^{\epsilon'},$$

and scaling by a quantity much smaller or much larger does not produce non-trivial limits.

Discrete Asymmetry

One would like to relate the potential scaling limits to SLE(6).
But:

Proposition

Consider a box of size σ_N centered on the line $y = -2\sigma_N$: it contains $\approx N$ sites of the front, but

$$\#\textit{black sites} - \#\textit{white sites} \approx N^{4/7} \gg \sqrt{N}.$$

And in fact, in the scaling limit, the law of the front will be singular with respect to SLE(6). This is related to *off-critical* percolation (*cf* Wendelin's talk).

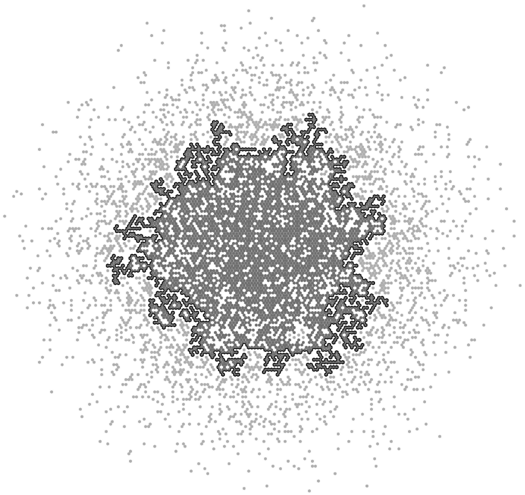
Application: geometry of diffusion fronts

Description of the model

We start at time $t = 0$ with a large number n of particles located at the origin, and we let them perform independent random walks.

At each time t, we then look at the sites containing at least one particle. These occupied sites can be regrouped into connected components, or "clusters", by connecting two occupied sites if they are adjacent on the lattice.

Description of the model



Different regimes

As time t increases, different regimes arise:

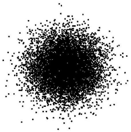
- At first, a very dense cluster around the origin forms. This clusters grows as long as t remains small compared to n.
- When t gets comparable to n, the cluster first continues to grow up to some time $t_{\max} = \lambda_{\max} n$.
- It then starts to decrease and it finally dislocates at some critical time $t_c = \lambda_c n$ - and never re-appears.

Remark: $t_c / t_{\max} = e$ is universal.

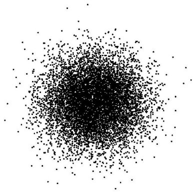
Evolution for $n = 10000$ particles: $t = 10$

Evolution for $n = 10000$ particles: $t = 100$

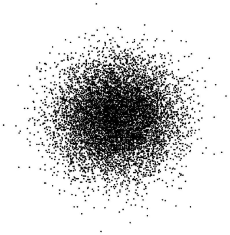
Evolution for $n = 10000$ particles: $t = 500$



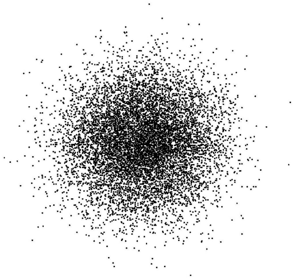
Evolution for $n = 10000$ particles: $t = 1000$



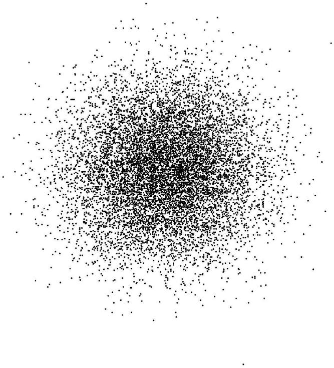
Evolution for $n = 10000$ particles: $t = 1463 = \lambda_{\max} n$



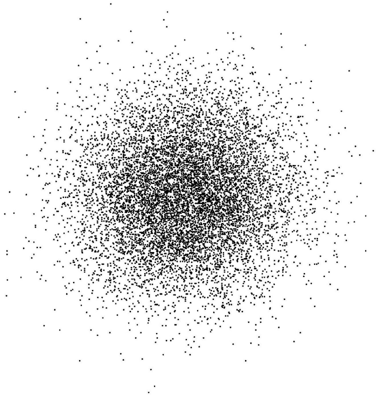
Evolution for $n = 10000$ particles: $t = 2500$



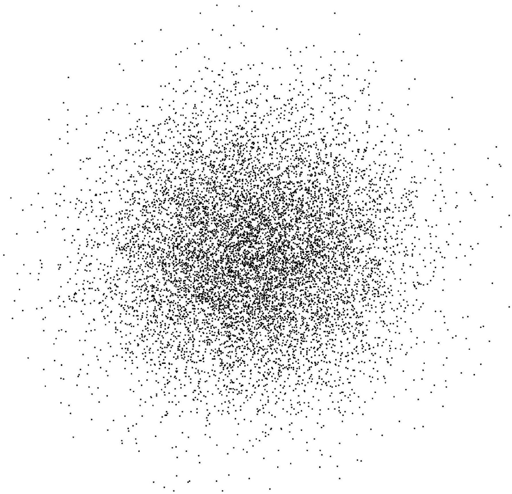
Evolution for $n = 10000$ particles: $t = 3977 = \lambda_c n$



Evolution for $n = 10000$ particles: $t = 5000$



Evolution for $n = 10000$ particles: $t = 10000$



Main ingredients

We first need a strong form of the Local Central Limit Theorem. The distribution of a simple random walk after t steps satisfies:

$$\pi_t(z) = \frac{\sqrt{3}}{2\pi t} e^{-\|z\|^2/t} + O\left(\frac{1}{t^2}\right).$$

($\sqrt{3}/2$ comes from the "density" of sites on the triangular lattice).

Main ingredients

The probability of occupation for a site z is

$$1 - (1 - \pi_t(z))^n \simeq 1 - e^{-n\pi_t(z)}.$$

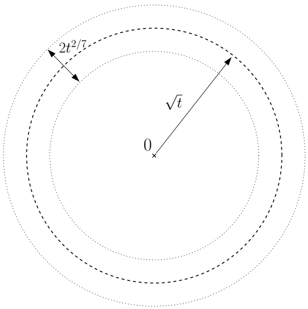
This is equal to $1/2$ for

$$r^*_{n,t} = \sqrt{t \log \frac{\lambda_c}{t/n}}$$

if $t \leq \lambda_c n$, with $\lambda_c = \sqrt{3}/2\pi \log 2$ (and it remains $< 1/2$ otherwise).

Main ingredients

An approximation using Gradient percolation is valid. The boundary remains localized in an annulus of width $\approx (\sqrt{t})^{4/7} = t^{2/7}$ around $r = r^* \asymp \sqrt{t}$. They represent a negligible fraction of the sites, we can thus use a *Poissonian approximation*.



Results: case $\lambda < \lambda_c$

Theorem (N.)

Consider $t_n = \lambda n$, with $\lambda < \lambda_c$. Then with probability tending to 1,

- *There exists a unique macroscopic interface surrounding 0.*
- *It remains localized in the annulus of width $\approx t^{2/7}$ around $r = r^* \asymp \sqrt{t}$.*
- *Its length behaves like $t^{5/7}$ and its roughness can be described via the universal exponent $7/4$ (it is locally of Hausdorff dimension $7/4$).*

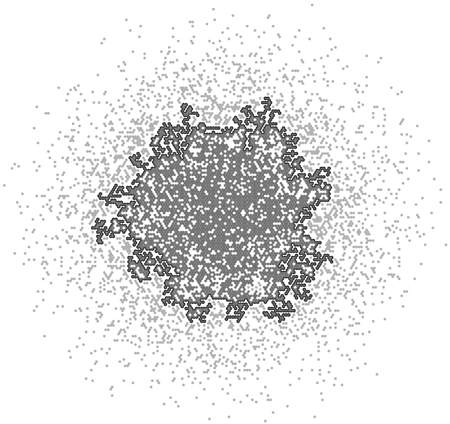
Remark: case $t \ll n$

Similar results are valid in the case $t = n^\alpha$, $\alpha < 1$.

Only the **transition window** (from parameter $1/2 + \epsilon$ to $1/2 - \epsilon$) is different. This window is of size $\asymp \sqrt{t}/\sqrt{\log t}$ around $r^* \asymp \sqrt{t \log t}$ (localized transition).

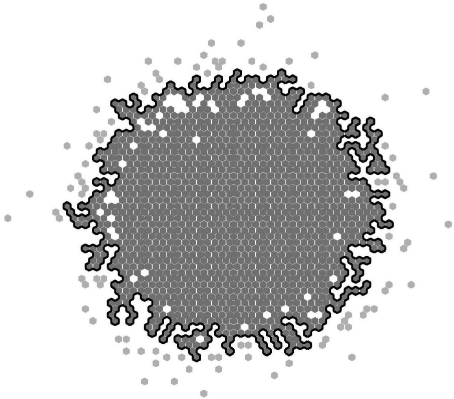
Remark: case $t \ll n$

When $t \asymp n$, gradual transition:



Remark: case $t \ll n$

When $t \ll n$, abrupt transition:



Results: case $\lambda > \lambda_c$

In the case $t_n = \lambda n$ with $\lambda > \lambda_c$, the whole picture can be "dominated" by a sub-critical percolation.

Hence for some constant $c = c(\lambda)$,

$$\mathbb{P}(\text{every cluster is of size} \leq c \log n) \to 1.$$

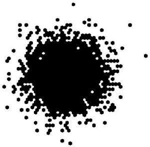
Model with a source

We now consider a model with a Poissonian source at the origin:

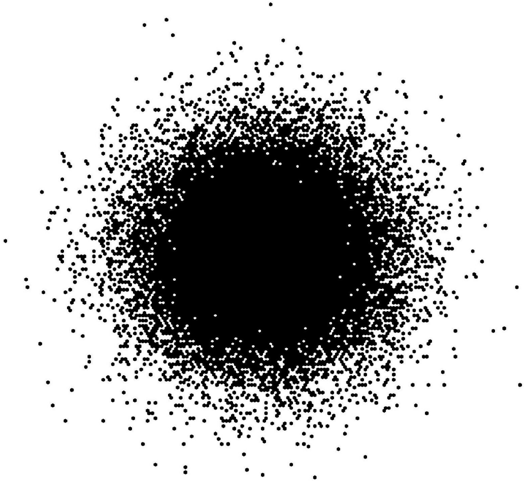
- Particles arrive at rate $\mu > 0$.
- Once arrived, they perform independent random walks.

Model with a source ($\mu = 50$): $t = 10$

Model with a source ($\mu = 50$): $t = 100$



Model with a source ($\mu = 50$): $t = 1000$



Model with a source

The occupation parameter is now

$$1 - e^{-\mu\rho_t(z)},$$

with

$$\rho_t(z) = \pi_0(z) + \ldots + \pi_t(z) \simeq \frac{\sqrt{3}}{2\pi} \int_{\|z\|^2/t}^{+\infty} \frac{e^{-u}}{u} du.$$

Model with a source

The behavior is then the same as previously for any $\mu > 0$ (no phase transition):

- There exists a unique macroscopic interface surrounding 0.
- It remains localized in the annulus of width $\approx t^{2/7}$ around $r = r^* \asymp \sqrt{t}$.
- Its length behaves like $t^{5/7}$ and it is locally of Hausdorff dimension $7/4$.

End

Thank you !