On the diameter of Gaussian free field excursion clusters away from criticality

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Joint work with Subhajit Goswami and Pierre-François Rodriguez

The model

Gaussian Free Field: Let $(\varphi_x : x \in \mathbb{Z}^d)$, $d \geq 3$, be the centered Gaussian field with covariances given by

$$\mathbb{E}[\varphi_x \varphi_y] = g(x, y),$$

$$g(x, y) := \sum_{n \geq 0} \mathbf{P}_x[X_n = y] \;\; (\asymp |x - y|^{-(d-2)}).$$

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Or alternatively:

$$d\mathbb{P}(\varphi) \propto \exp\Big[-\frac{1}{2}\mathcal{E}(\varphi,\varphi)\Big]\, d\varphi,$$

$$\mathcal{E}(f,f) := \frac{1}{4d} \sum_{\substack{x,y \in \mathbb{Z}^d \\ x \sim y}} (f(y) - f(x))^2.$$

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Level-set percolation: For each fixed $h \in \mathbb{R}$, consider the induced (random) subgraph of $\mathbb{Z}^d$ with vertex set

$$\{\varphi \geq h\} := \{x \in \mathbb{Z}^d : \ \varphi_x \geq h\}.$$

Existence of phase transition

We wish to study the connective properties of $\{\varphi \geq h\}$ as h varies.

The **critical parameter** for percolation is defined as

$$h_* = h_*(d) := \inf\{h \in \mathbb{R} : \ \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \infty] = 0\}.$$

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The phase transition is non-trivial for any $d \geq 3$:

- $h_*(d) \geq 0 \ \ \forall d \geq 3$ (Bricmont–Lebowitz–Maes'87)

  Actually $h_*(d) > 0 \ \ \forall d \geq 3$ (Drewitz–Prévost–Rodriguez'18)
- $h_*(3) < +\infty$ (Bricmont–Lebowitz–Maes'87)
- $h_*(d) < +\infty \ \ \forall d \geq 3$ (Rodriguez–Sznitman'13)

Sharpness of the phase transition

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- If $h > h_*(d)$, then for $\beta = \beta(d) \in (0,1]$, $c = c(h,d) > 0$

$$\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N] \leq e^{-c\,N^{\beta}}.$$

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- If $h < h_*(d)$, then for $\beta = \beta(d) \in (0,1]$, $c = c(h,d) > 0$

$$\mathbb{P}\left[\begin{array}{c}\text{there is a } \textbf{unique} \text{ macroscopic} \\ \text{cluster of } \{\varphi \geq h\} \text{ inside } B_N\end{array}\right] \geq 1 - e^{-c\, N^{\beta}}.$$

The diameter of a finite cluster

As a simple consequence of sharpness for every $h \neq h_*$, the *(truncated)* one arm probability decays at least stretched exponentially fast:

$$\mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N,\, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] \leq e^{-cN^{\beta}}. \qquad (\star)$$

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Theorem (Popov–Teixeira, Popov–Rath '15)

For $d \geq 4$ and $h > h_*$

$$e^{-C(h)N} \leq \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N] \leq e^{-c(h)N}.$$

For $d = 3$, $h > h_*$ and any $\varepsilon > 0$

$$e^{-C(h)N} \leq \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N] \leq \exp\Big[-c(h,\varepsilon)\frac{N}{(\log N)^{3+\varepsilon}}\Big].$$

Our result

Theorem (Goswami–Rodriguez–S. '20)

If $\mathbf{d} = \mathbf{3}$*, then for every* $h \neq h_*$*,*

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \xleftrightarrow{\varphi \geq h} \partial B_N,\ 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] = -\frac{\pi}{3}(h - h_*)^2. \qquad (1)$$

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Remark 1: As mentioned before, the only previously known case was $d \geq 4$ and $h > h_*$.

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Remark 1: As mentioned before, the only previously known case was $d \geq 4$ and $h > h_*$.

Remark 2: A result analogous to the case $d = 3$ above for independent percolation is currently unavailable. Actually, that would allow to compute the critical exponent ν for the correlation length.

Capacity

For a finite subset $K \subset \mathbb{Z}^d$, its *capacity* is defined as follows

$$\mathrm{cap}(K) := \sum_{x \in \partial_{int} K} \mathbf{P}_x[X_n \notin K \quad \forall n \geq 1].$$

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Some examples:

$$\mathrm{cap}(B_N) \sim \frac{\mathrm{cap}([0,1]^d)}{d} N^{d-2} \quad \forall d \geq 3,$$

$$\mathrm{cap}\big([0,N] \times \{0\}^{d-1}\big) \sim \begin{cases} \frac{2\pi}{3} \frac{N}{\log N}, & \text{if } d = 3 \\ C_d N, & \text{if } d \geq 4. \end{cases}$$

Note the similarity between the our large deviation results and the above asymptotic for $\mathrm{cap}\big([0,N] \times \{0\}^{d-1}\big)$.

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Note the similarity between the our large deviation results and the above asymptotic for $\mathrm{cap}\big([0,N] \times \{0\}^{d-1}\big)$. Before explaining this connection, let us see some other examples of large deviation results for the GFF.

Large deviation and entropic repulsion

Theorem (Bolthausen–Deuschel–Zeitouni '95)

Let $B_N := \{x \in \mathbb{Z}^d : |x|_\infty \leq N\}$, *then*

$$\lim_N \frac{1}{N^{d-2}\log N} \log \mathbb{P}\big[\varphi_x \geq 0 \textit{ for all } x \in B_N\big] = -\frac{2}{d} g(0) \mathrm{cap}([0,1]^d).$$

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Furthermore, conditionally on the event $\{\varphi_x \ge 0 \textit{ for all } x \in B_N\}$, *one has*

$$\text{“ } \varphi_x \sim 2\sqrt{g(0)\log N} \quad \textit{w.h.p.} \quad \textit{on } B_N \text{ ”}$$

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Remark: Intuitively, the theorems says that the best strategy for φ to realize the event $\{\varphi_x \geq 0$ for all $x \in B_N\}$ is through a shift of order $2\sqrt{g(0)\log N}$ inside B_N. This phenomenon is called *"entropic repulsion"*.

Large deviation and entropic repulsion

Theorem (Sznitman '15)

For every $h < h_*$,

$$\lim_N \frac{1}{N^{d-2}} \log \mathbb{P}\big[B_N \overset{\varphi \geq h}{\not\longleftrightarrow} \partial B_{2N}\big] = -\frac{1}{2d}(h_* - h)^2 \mathrm{cap}([0,1]^d).$$

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Theorem (Nitzschner '18)

For every non-empty compact set $A \subset [0,1]^d \subset \mathbb{R}^d$ and $h < h_*$,

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where $A_N := \{x \in \mathbb{Z}^d : \; x/N \in A\}$.

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Remark: In these results, lower and upper bounds were expressed in terms of alternative critical parameters h_{**} and $\bar{h}$, which were later proved to be equal due to sharpness.

Entropic lower bound

- If $\tilde{\mathbb{P}}$ is absolutely continuous w.r.t. $\mathbb{P}$, then

$$\mathbb{P}[A] \geq \tilde{\mathbb{P}}[A] \exp\left[-\frac{1}{\tilde{\mathbb{P}}[A]}\left(H(\tilde{\mathbb{P}}|\mathbb{P}) + e^{-1}\right)\right],$$

where $H(\tilde{\mathbb{P}}|\mathbb{P}) := \tilde{\mathbb{E}}\left[\log \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right]$ is the relative entropy.

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- Given $f : \mathbb{Z}^d \to \mathbb{R}$ with bounded support, let

$$d\tilde{\mathbb{P}}(\varphi) := \exp\left[\mathcal{E}(f,\varphi) - \frac{1}{2}\mathcal{E}(f,f)\right] d\mathbb{P}(\varphi).$$

One can easily verify that

  - φ under $\tilde{\mathbb{P}}$ has the same law as $\varphi + f$ under $\mathbb{P}$,
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- The following variational principle is valid

$$\inf_{f \equiv \alpha \text{ on } K} \mathcal{E}(f,f) = \alpha^2 \mathrm{cap}(K).$$

Lower bound for $d = 3$ and $h > h_*$

Proposition (Entropic lower bound)

Let $K_N \subset \mathbb{Z}^d$ and $A_N \in \{0,1\}^{K_N}$ be two sequences of domains and events such that $\mathrm{cap}(K_N) \to \infty$ and, for every $h \in J \subset \mathbb{R}$,

$$\mathbb{P}[\{\varphi \geq h\} \in A_N] \to 1.$$

Then for every $h \notin J$,

$$\liminf_N \frac{1}{\mathrm{cap}(K_N)} \log \mathbb{P}\big[\{\varphi \geq h\} \in A_N\big] \geq -\frac{1}{2} d(h, J)^2.$$

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Proof of the lower bound for $d = 3$ and $h > h_*$: Apply the above proposition for $J = (-\infty, h_*)$ and A_N being the existence of a long crossing inside the "tube" K_N of length N and width $(\log N)^C$. Finally notice that

$$\mathrm{cap}(K_N) \sim \mathrm{cap}\big([0, N] \times \{0\}^2\big) \sim \frac{2\pi}{3} \frac{N}{\log N}.$$

□

An upper bound for harmonic deviations

Markov decomposition of the GFF: For any domain (a box, say) U, one can decompose φ as a sum of two *independent* fields

$$\varphi = h_U + \psi_U,$$

where $h_U(x) := E_x[\varphi_{X_{H_{U^c}}}]$ is the harmonic average of φ in U and ψ_U is the local field in U.

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Proposition (Sznitman '15)

For every $\alpha > 0$

$$\limsup_{L} \sup_{\mathcal{C}} \frac{1}{\mathrm{cap}(\mathcal{C})} \log \mathbb{P}\Big[\bigcap_{B \in \mathcal{C}} \big\{ \sup_{x \in B} h_{U_B}(x) \geq \alpha \big\} \Big] \leq -\frac{1}{2}\alpha^2,$$

where the supremum on $\mathcal{C}$ runs over "nice" families of "distant" L-boxes B and U_B is some big box containing B.

Remark: The proof of this proposition is based on the Borell-TIS inequality.

Upper bound for $d = 3$ and $h > h_*$

We use a *"two scale argument"*:

- Let $L = (\log N)^{1+\delta}$ and say that an L-box B is φ-bad if it is crossed in $\{\varphi \geq h\}$.

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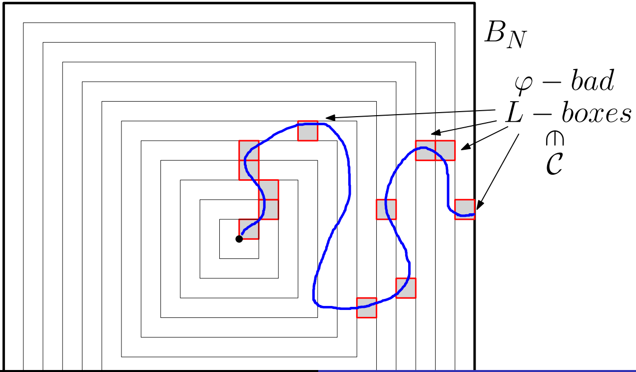
- Let $L = (\log N)^{1+\delta}$ and say that an L-box B is φ-bad if it is crossed in $\{\varphi \geq h\}$.
- Any path from 0 to ∂B_N in $\{\varphi \geq h\}$ induces a family $\mathcal{C}$ of L-boxes which are "nice", "distant" and φ-bad. There are at most $C^{N/L}$ such $\mathcal{C}$'s.



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- Note that a φ-bad box is either ψ-bad (i.e. crossed in $\{\psi_U \geq h_* + \varepsilon\}$) or it is h-bad (i.e. $\left\{ \inf_{x \in B} h_U(x) \geq \alpha \right\}$ happens with $\alpha = h - h_* - \varepsilon$).

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- Bound the cost of ψ-bad boxes by independence and the a priori stretched exponential bound for the one arm probability.
- Bound the cost of h-bad boxes using the Proposition from last slide. Finally, lower bound the capacity of all $\mathcal{C}$'s:

$$\mathrm{cap}(\mathcal{C}) \geq (1 + o(1))\, \mathrm{cap}([0, N] \times \{0\}^2) \sim \frac{2\pi}{3} \frac{N}{\log N}.$$

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Summing up:

$$\mathbb{P}[0 \overset{\varphi \geq h}{\longleftrightarrow} \partial B_N] \leq C^{N/L} \left[\exp\left(-c(\varepsilon) L^{\beta} \frac{N}{L} \right) + \exp\left(-\frac{\pi + o(1)}{3} (h - h_* - \varepsilon)^2 \frac{N}{\log N} \right) \right].$$

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Just recall that $L = (\log N)^{1+\delta}$, choose $\delta < \frac{\beta}{1-\beta}$ and let $\varepsilon \to 0$. □

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  $$-\inf \left\{ \frac{1}{2d} \int_{\mathbb{R}^d} | \bigtriangledown f|^2 \;:\; f \in C_0^\infty \text{ and } \int_{D(f,h)} \theta(h-f) = 1 \right\},$$

  where $D(f,h)$ is the connected component of $\{x \in \mathbb{R}^d : h - f(x) \leq h_*\}$ containing the origin 0.

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$$\lim_N \frac{1}{N^{(d-2)/d}} \log \mathbb{P}[|\mathcal{C}_0^h| = N] =$$

$$-\inf\left\{ \frac{1}{2d}\int_{\mathbb{R}^d} |\nabla f|^2 \;:\; f \in C_0^\infty \text{ and } \int_{D(f,h)} \theta(h-f) = 1 \right\},$$

  where $D(f,h)$ is the connected component of $\{x \in \mathbb{R}^d : h - f(x) \leq h_*\}$ containing the origin 0. For the optimal function f, the profile $\theta(h-f)$ and the domain $D(f,h)$ should represent, respectively, the typical density and shape of $\mathcal{C}_0^h$ when conditioned on $\{|\mathcal{C}_0^h| = N\}$.

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- Obtain similar results for the *volume* of a finite cluster.

  Conjecturally, for $h \neq h_*$:

  $$\lim_N \frac{1}{N^{(d-2)/d}} \log \mathbb{P}[|\mathcal{C}_0^h| = N] =$$
  $$-\inf\left\{ \frac{1}{2d} \int_{\mathbb{R}^d} |\bigtriangledown f|^2 \ : \ f \in C_0^\infty \text{ and } \int_{D(f,h)} \theta(h-f) = 1 \right\},$$

  where $D(f,h)$ is the connected component of $\{x \in \mathbb{R}^d : h - f(x) \leq h_*\}$ containing the origin 0. For the optimal function f, the profile $\theta(h-f)$ and the domain $D(f,h)$ should represent, respectively, the typical density and shape of $\mathcal{C}_0^h$ when conditioned on $\{|\mathcal{C}_0^h| = N\}$.
- Study the model at and near the critical point h_*. This should be more treatable on the cable system.

Thank you!