CONEXÕES ENTRE

SISTEMAS DINÂMICOS NÃO LINEARES E ENTROPIA

QUANDO O MAXIMO EXPOENTE DE LYAPUNOV É ZERO

Constantino Tsallis

Centro Brasileiro de Pesquisas Físicas

and National Institute of Science and Technology for Complex Systems

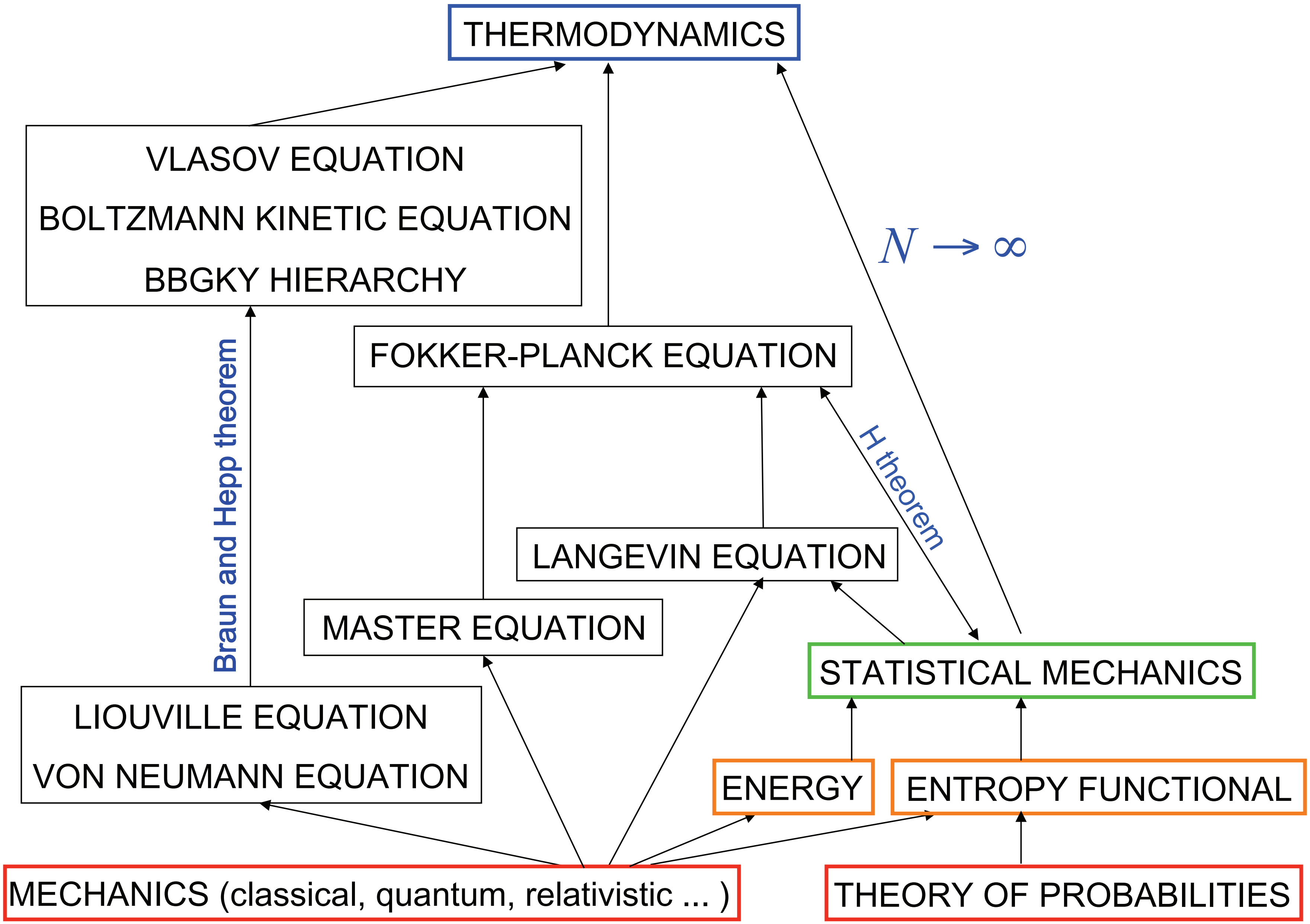
Rio de Janeiro

and

Santa Fe Institute, New Mexico, USA

SANTA FE INSTITUTE

UFRJ - Matemática, Junho 2011



THERMODYNAMICS
VLASOV EQUATION
BOLTZMANN KINETIC EQUATION
BBGKY HIERARCHY
$N \to \infty$
FOKKER-PLANCK EQUATION
Braun and Hepp theorem
H theorem
LANGEVIN EQUATION
MASTER EQUATION
STATISTICAL MECHANICS
LIOUVILLE EQUATION
VON NEUMANN EQUATION
ENERGY
ENTROPY FUNCTIONAL
MECHANICS (classical, quantum, relativistic ...)
THEORY OF PROBABILITIES

Enrico FERMI *Thermodynamics* (Dover, 1936)

The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.

POSTULATE FOR THE ENTROPIC FUNCTIONAL

	$p_i = \frac{1}{W} \quad (\forall i)$ **equiprobability**	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1\right)$
BG entropy (*q* =*1*)	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Entropy *Sq* (*q real*)	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

additive

Concave

Extensive

Lesche-stable

Finite entropy production per unit time

Pesin-like identity (with largest entropy production)

Composable

Topsoe-factorizable

nonadditive (if $q \neq 1$)

Possible generalization of Boltzmann-Gibbs statistical mechanics

[C.T., J. Stat. Phys. **52**, 479 (1988)]

DEFINITIONS : $q-logarithm:$ $\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0;\ \ln_1 x = \ln x)$

$q-exponential:$ $e_q^x \equiv [1+(1-q)\,x]^{\frac{1}{1-q}} \quad (e_1^x = e^x)$

Hence, the entropies can be rewritten :

	equal probabilities	*generic probabilities*
BG entropy $(q=1)$	$k\ \ln W$	$k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$
entropy S_q $(q \in R)$	$k\ \ln_q W$	$k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$

TYPICAL SIMPLE SYSTEMS:

$$\text{e.g.,}\quad W(N) \propto \square^{N} \quad (\square > 1)$$

Short-range space-time correlations

Markovian processes (short memory), Additive noise

Strong chaos (positive maximal Lyapunov exponent), Ergodic, Euclidean geometry

Short-range many-body interactions, weakly quantum-entangled subsystems

Linear and homogeneous Fokker-Planck equations, Gausssians

→ **Boltzmann-Gibbs entropy (additive)**

→ **Exponential dependences (Boltzmann-Gibbs weight, ...)**

TYPICAL COMPLEX SYSTEMS:

$$\text{e.g.,}\quad W(N) \propto N^{\rho} \quad (\rho > 0)$$

Long-range space-time correlations

Non-Markovian processes (long memory), Additive and multiplicative noises

Weak chaos (zero maximal Lyapunov exponent), Nonergodic, Multifractal geometry

Long-range many-body interactions, strongly quantum-entangled sybsystems

Nonlinear and/or inhomogeneous Fokker-Planck equations, *q*-Gaussians

→ **Entropy S*q* (nonadditive)**

→ ***q*-exponential dependences (asymptotic power-laws)**

- **Additive *versus* Extensive**

- Nonlinear dynamical systems

- Central Limit Theorem

- Predictions, verifications, applications

ADDITIVITY: O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), page 167

An entropy is additive if, for any two probabilistically independent systems A and B,

$$S(A+B)=S(A)+S(B)$$

Therefore, since

$$S_q(A+B)=S_q(A)+S_q(B)+(1-q)\,S_q(A)\,S_q(B)\ ,$$

S_{BG} and $S_q^{Renyi}\,(\forall q)$ are additive, and $S_q\ \ (\forall q\neq 1)$ is nonadditive .

EXTENSIVITY:

Consider a system $\Sigma \equiv A_1 + A_2 + \ldots + A_N$ made of N (not necessarily independent) identical elements or subsystems A_1 and A_2, ..., A_N.

An entropy is extensive if

$$0 < \lim_{N\to\infty} \frac{S(N)}{N} < \infty\ ,\ i.e.,\ \ S(N) \propto N\ \ (N\to\infty)$$

TYPICAL SIMPLE SYSTEMS (equal probabilities):

$$W(N) \propto \square^{N} \quad (N \to \infty \; ; \; \square > 1)$$

$$\Rightarrow S_{BG}(N) = k_B \ln W(N) \propto N \qquad (EXTENSIVE!)$$

TYPICAL COMPLEX SYSTEMS (equal probabilities):

$$W(N) \propto N^{\rho} \quad (N \to \infty \; ; \; \rho > 0) \qquad << \square^{N}$$

$$\Rightarrow S_q(N) = k_B \ln_q W(N) = k_B \frac{[W(N)]^{1-q} - 1}{1-q}$$

$$\propto N^{\rho(1-q)} \quad \left[\text{if } q = 1 - \frac{1}{\rho} \right] \propto N \qquad (EXTENSIVE!)$$

HYBRID PASCAL - LEIBNITZ TRIANGLE

(N=0) $1\times\frac{1}{1}$

(N=1) $1\times\frac{1}{2}$ $1\times\frac{1}{2}$

(N=2) $1\times\frac{1}{3}$ $2\times\frac{1}{6}$ $1\times\frac{1}{3}$

(N=3) $1\times\frac{1}{4}$ $3\times\frac{1}{12}$ $3\times\frac{1}{12}$ $1\times\frac{1}{4}$

(N=4) $1\times\frac{1}{5}$ $4\times\frac{1}{20}$ $6\times\frac{1}{30}$ $4\times\frac{1}{20}$ $1\times\frac{1}{5}$

(N=5) $1\times\frac{1}{6}$ $5\times\frac{1}{30}$ $10\times\frac{1}{60}$ $10\times\frac{1}{60}$ $5\times\frac{1}{30}$ $1\times\frac{1}{6}$

$$\sum_{n=0}^{N}\binom{N}{n}r_{N,n}=1 \quad (\forall\ N)$$

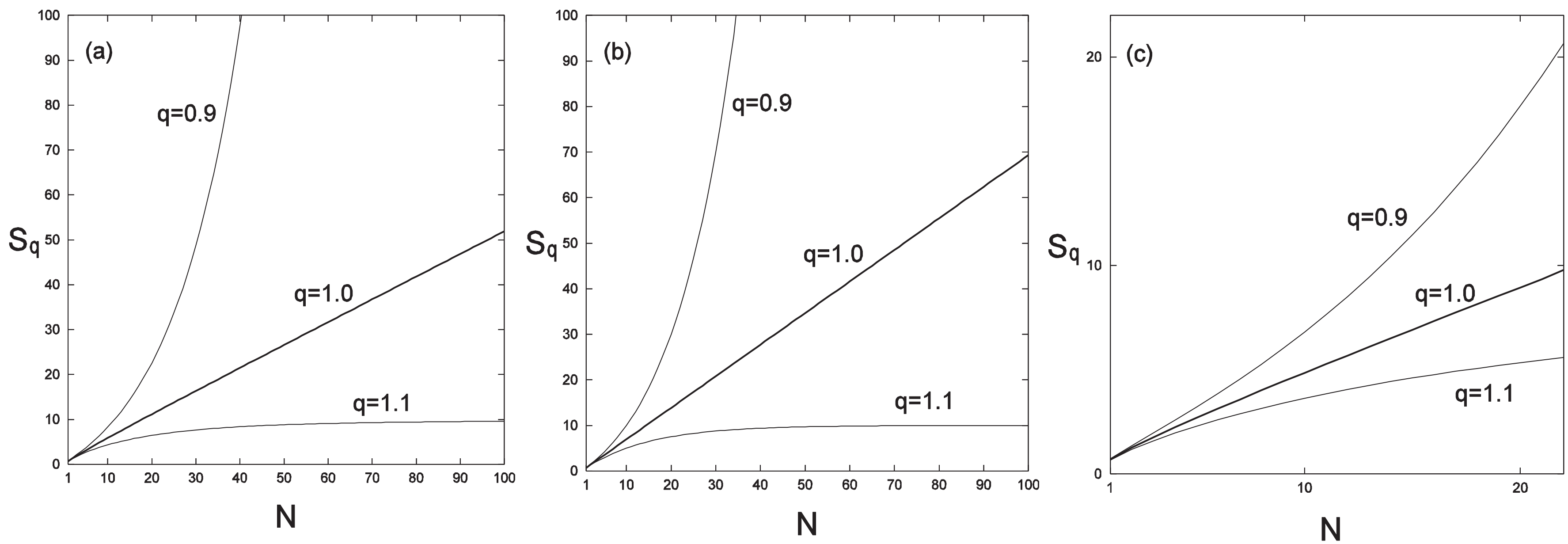
Blaise **Pascal** (1623-1662)
Gottfried Wilhelm **Leibnitz** (1646-1716)
Daniel **Bernoulli** (1700-1782)

$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N \quad (N \to \infty)$

I don't believe that atoms exist!

Ernst Mach (January 1897, Vienna)



Leibnitz triangle

$$\left(p_{N,0} = \frac{1}{N+1} \right)$$

N independent coins

$$\left(p_{N,0} = p^N \quad \text{with } p = 1/2 \right)$$

Stretched exponential

$$\left(p_{N,0} = p^{N^\alpha} \quad \text{with } p = \alpha = 1/2 \right)$$

(All three examples **strictly** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA **102**, 15377 (2005)

Asymptotically scale-invariant (d=2)

$$(N=0) \qquad 1$$

$$(N=1) \qquad 1/2 \quad 1/2$$

$$(N=2) \qquad 1/3 \quad 1/6 \quad 1/3$$

$$(N=3) \qquad 3/8 \quad 5/48 \quad 5/48 \quad 0$$

$$(N=4) \qquad 2/5 \quad 3/40 \quad 1/20 \quad 0 \quad 0$$

d+1

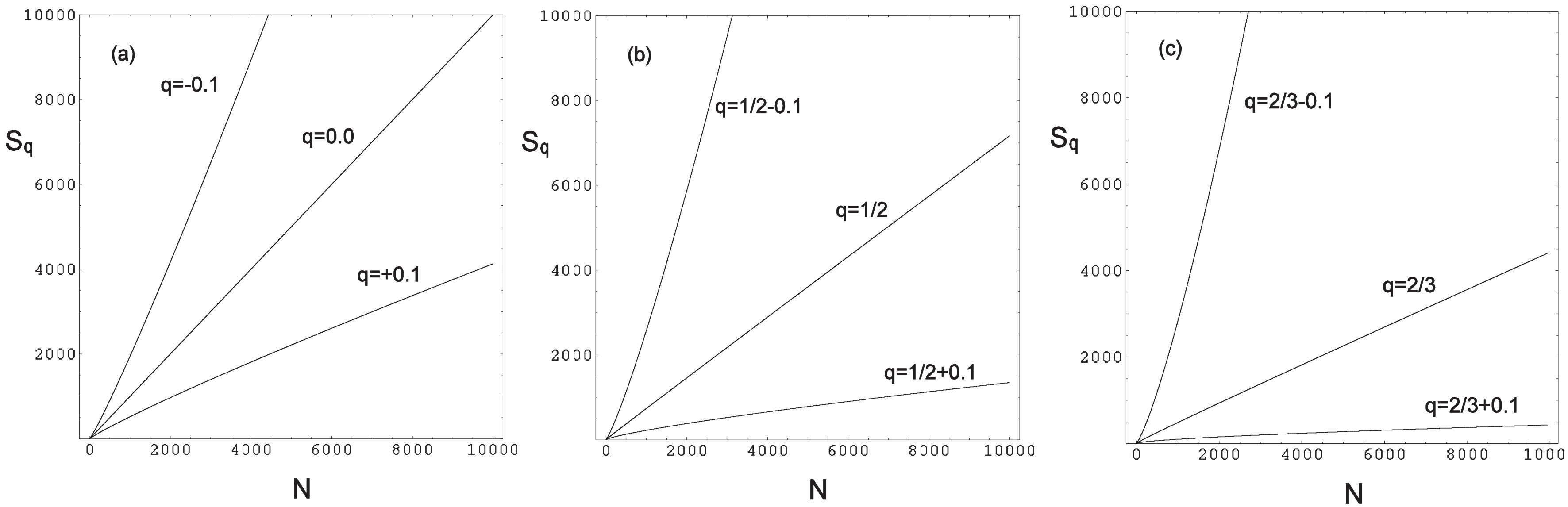
(It asymptotically satisfies the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA **102**, 15377 (2005)

$q \neq 1$ SYSTEMS

i.e., such that $S_q(N) \propto N \quad (N \to \infty)$

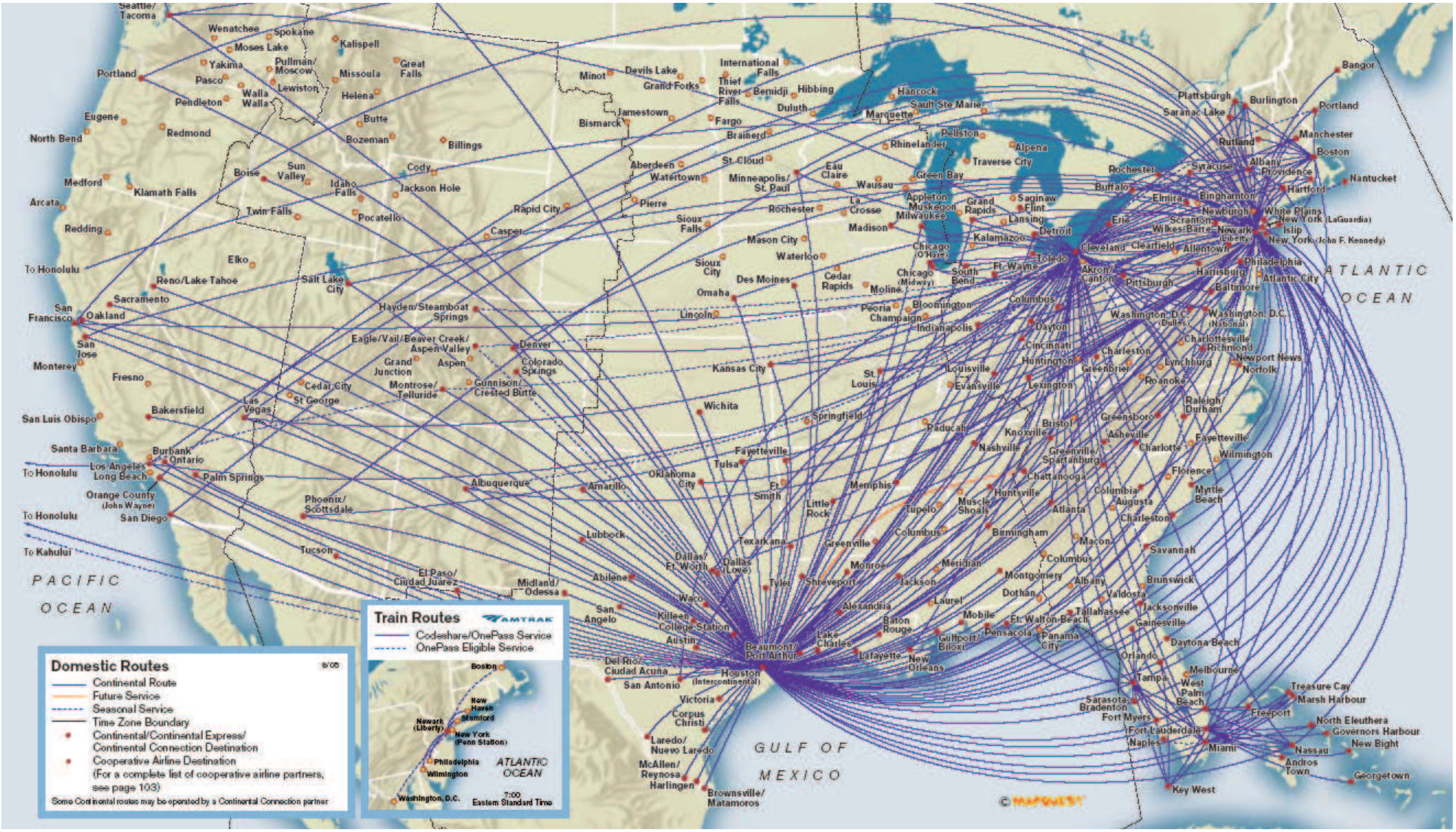
$(d=1)$ $(d=2)$ $(d=3)$



$$q = 1 - \frac{1}{d}$$

(All three examples **asymptotically** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA **102**, 15377 (2005)



Continental Airlines

PHYSICAL REVIEW E **78**, 021102 (2008)

Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

Filippo Caruso[1] and Constantino Tsallis[2,3]

[1]*NEST CNR-INFM and Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126 Pisa, Italy*

[2]*Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*

[3]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*

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The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size L) of some (much larger) d-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to L^{d-1}. Here we show, for $d=1,2$, that the (nonadditive) entropy S_q satisfies, for a special value of $q \neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. **102** 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index q.

SPIN ½ XY FERROMAGNET WITH TRANSVERSE MAGNETIC FIELD:

$$\hat{\mathcal{H}} = -\sum_{j=1}^{N-1} \left[(1+\gamma)\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + (1-\gamma)\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y + 2\lambda \hat{\sigma}_j^z\right]$$

$|\gamma| = 1 \quad \rightarrow$ *Ising ferromagnet*

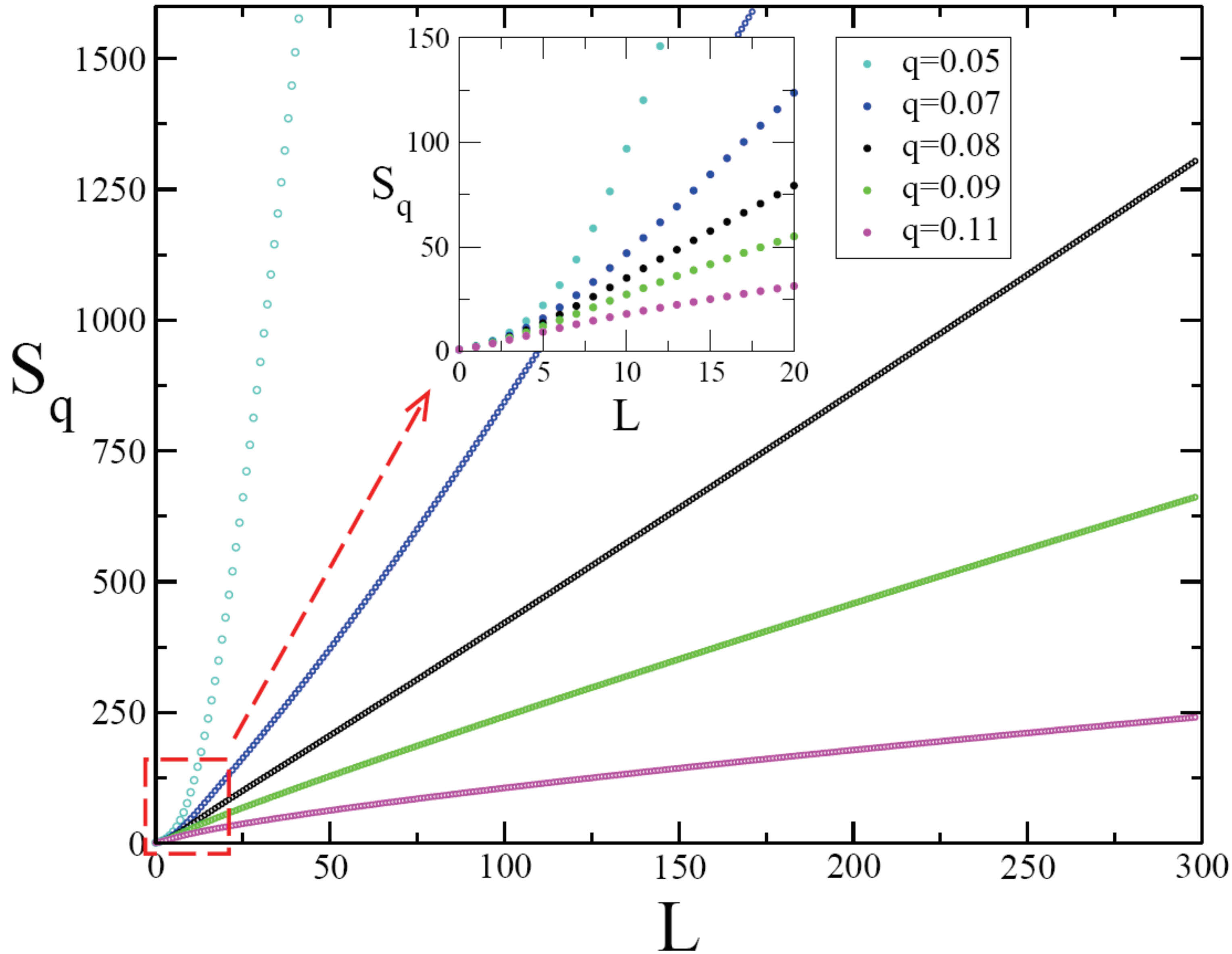
$0 < |\gamma| < 1 \quad \rightarrow$ *anisotropic XY ferromagnet*

$\gamma = 0 \quad \rightarrow$ *isotropic XY ferromagnet*

$\lambda \equiv$ *transverse magnetic field*

$L \equiv$ *length of a block within a* $N \rightarrow \infty$ *chain*

F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)



F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

Using a Quantum Field Theory result

in P. Calabrese and J. Cardy, JSTAT P06002 (2004)

we obtain, at the critical transverse magnetic field,

$$q_{ent} = \frac{\sqrt{9+c^2}-3}{c}$$

with $c \equiv$ *central charge in conformal field theory*

Hence

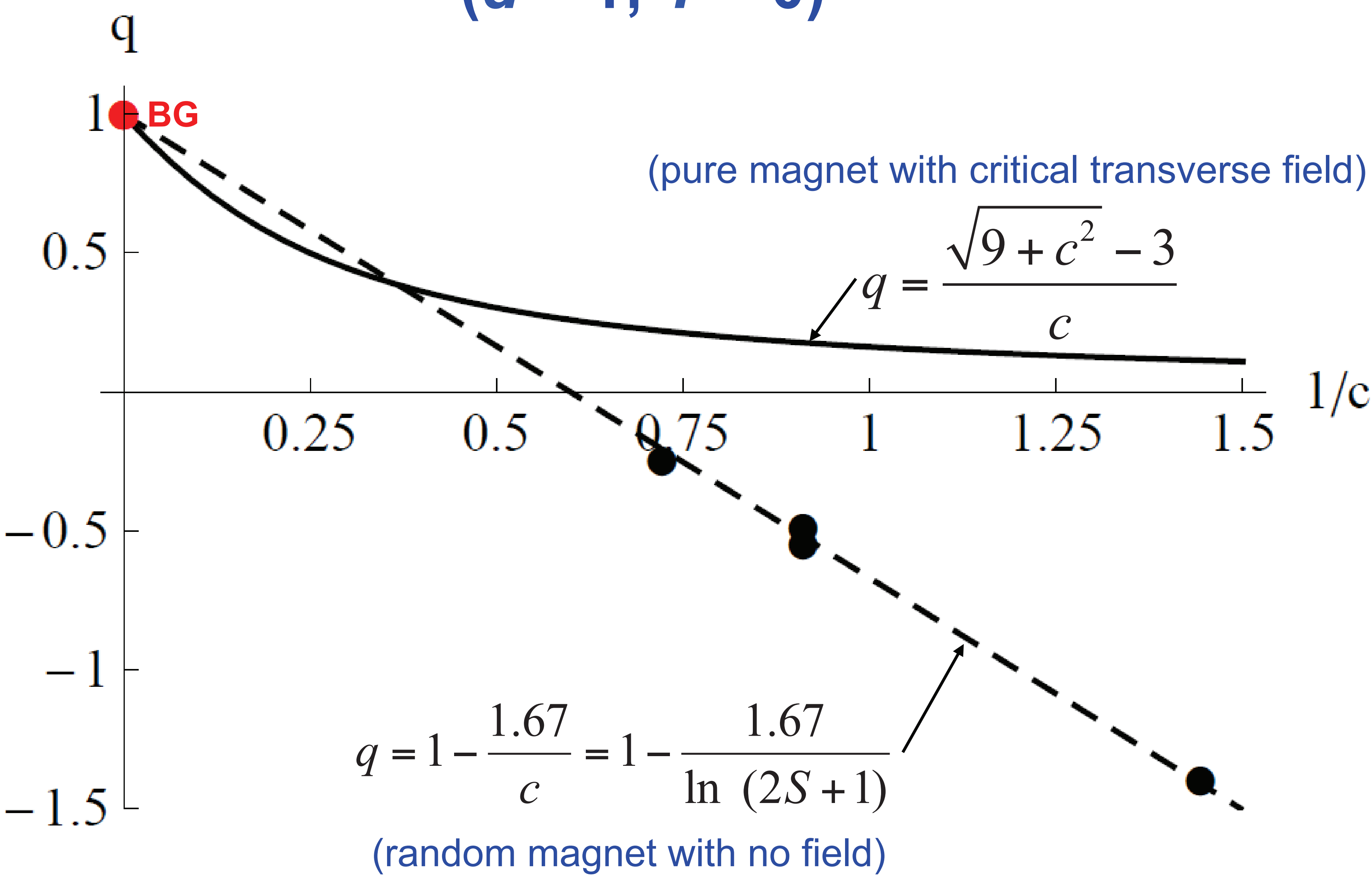
Ising and anisotropic XY ferromagnets $\Rightarrow\ c = \frac{1}{2}\ \Rightarrow\ q_{ent} = \sqrt{37}-6\ \approx 0.0828$

and

Isotropic XY ferromagnet $\Rightarrow\ c = 1\ \Rightarrow\ q_{ent} = \sqrt{10}-3\ \approx 0.1623$

F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

(d = 1; T = 0)



A Saguia and MS Sarandy, Phys Lett A **374**, 3384 (2010)

Summarizing, for a wide class of quantum systems or subsystems with N elements, we know that

$$
\begin{array}{llll}
S_{BG}(N) \propto \ln L & \propto \ln N & \neq N & \text{for } d=1 \text{ quantum chains} \\
\propto L & \propto \sqrt{N} & \neq N & \text{for } d=2 \text{ bosonic systems} \\
\propto L^2 & \propto N^{2/3} & \neq N & \text{for } d=3 \text{ black hole} \\
\propto L^{d-1} & \propto N^{(d-1)/d} & \neq N & \text{for } d\text{-dimensional bosonic systems} \\
 & & & (d>1;\ \text{area law})
\end{array}
$$

$$\propto \frac{L^{d-1}-1}{d-1} \equiv \ln_{2-d} L \neq L^d \propto N \qquad (d \geq 1) \qquad \text{(NONEXTENSIVE!)}$$

For the same class of quantum systems, we expect

$$S_{q_{ent}}(N) \propto L^d \propto N \qquad (d \geq 1;\ q_{ent} \neq 1) \qquad \text{(EXTENSIVE!)}$$

(analytically and/or computationally shown for $d = 1,2$)

F. Caruso and C. T., Phys Rev E **78,** 021101 (2008)

SYSTEMS	ENTROPY S_{BG} (additive)	ENTROPY S_q ($q<1$) (nonadditive)
Short-range interactions, weakly entangled blocks, etc	EXTENSIVE	NONEXTENSIVE
Long-range interactions (QSS), strongly entangled blocks, etc	NONEXTENSIVE	EXTENSIVE

quarks-gluons, plasma, curved space ...?

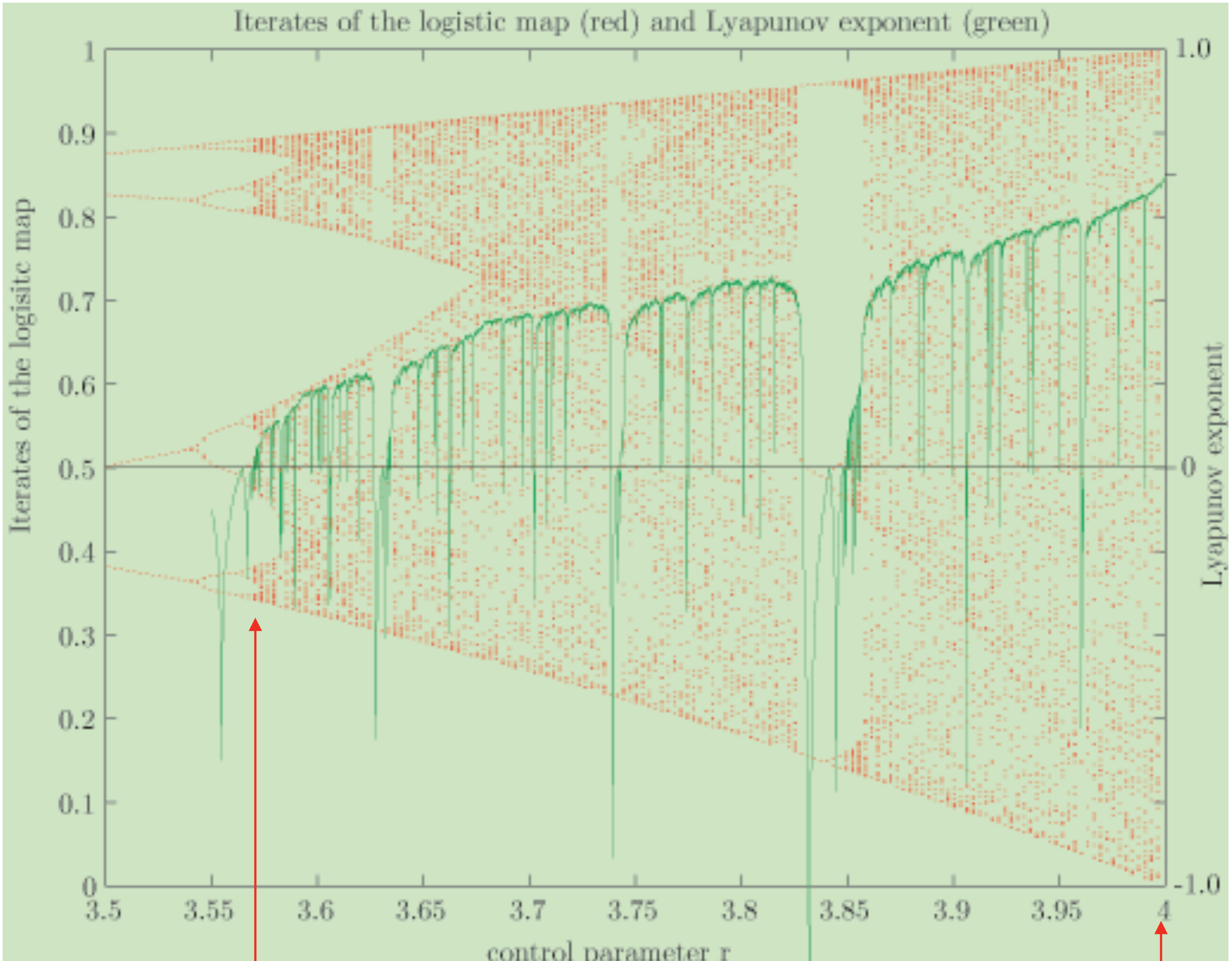
- Additive *versus* Extensive

- **Nonlinear dynamical systems**

- Central Limit Theorem

- Predictions, verifications, applications

$$x_{t+1} = 1 - a x_t^2 \quad (0 \le a \le 2)$$



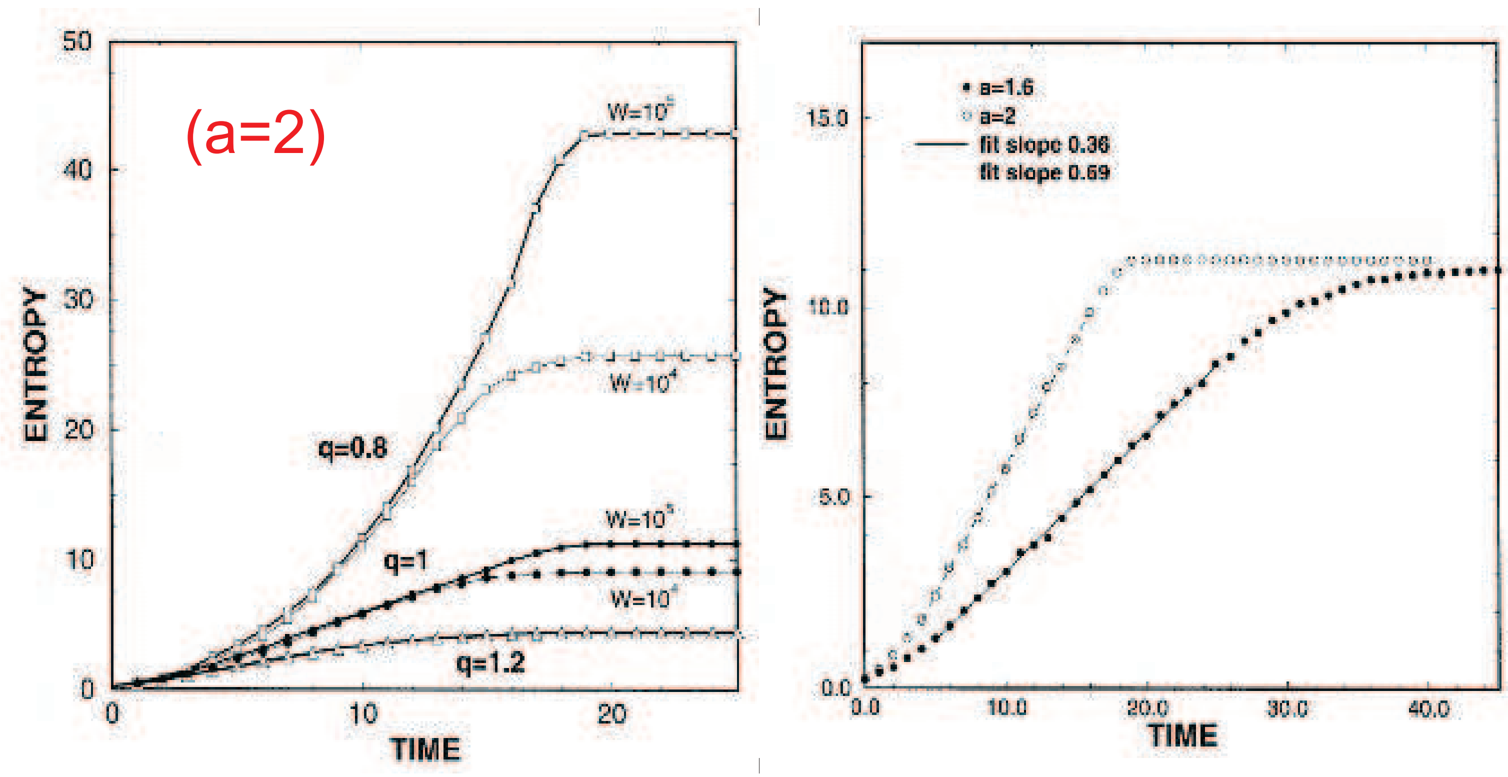
edge of chaos

fully developed chaos

LOGISTIC MAP:

$$x_{t+1} = 1 - a\,x_t^2 \quad (0 \le a \le 2;\ \ -1 \le x_t \le 1;\ t = 0,1,2,\ldots)$$

(strong chaos, i.e., **positive** Lyapunov exponent)



V. Latora, M. Baranger, A. Rapisarda and C. T., Phys. Lett. A **273**, 97 (2000)

We verify

$$K_1 = \lambda_1 \quad (Pesin-like\ identity)$$

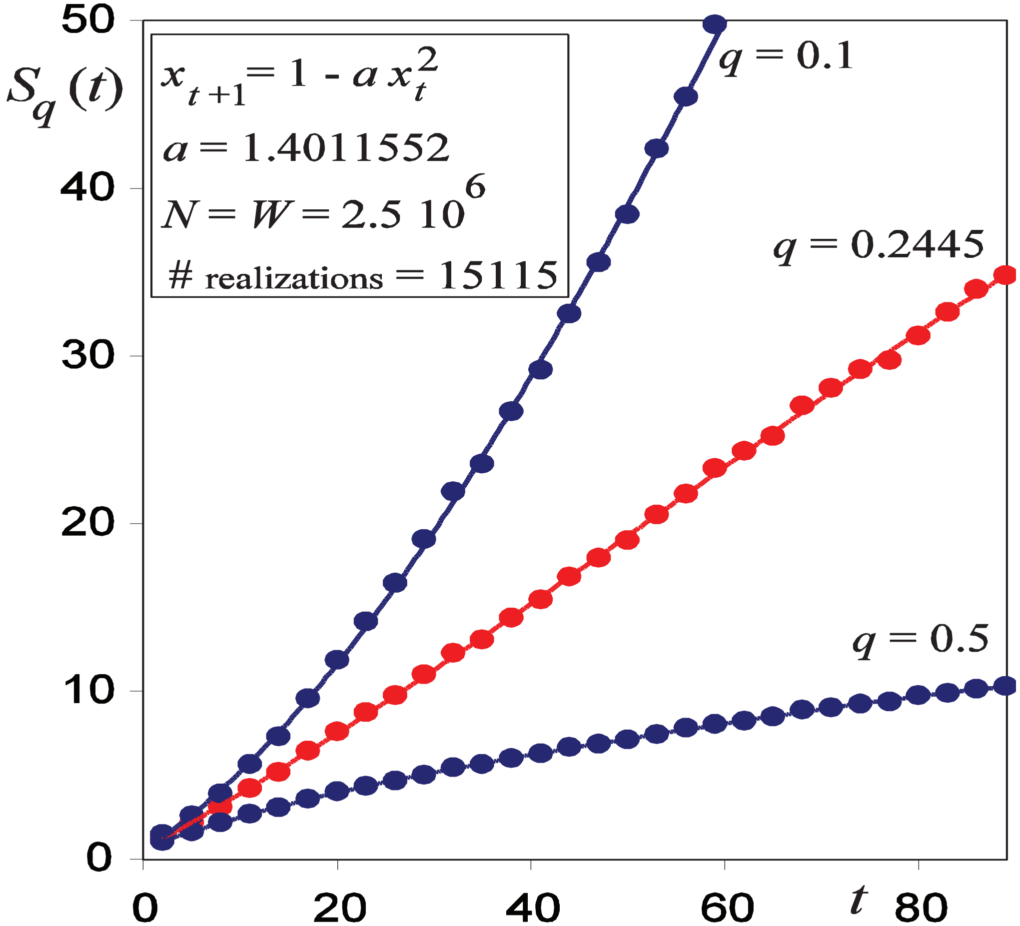
where

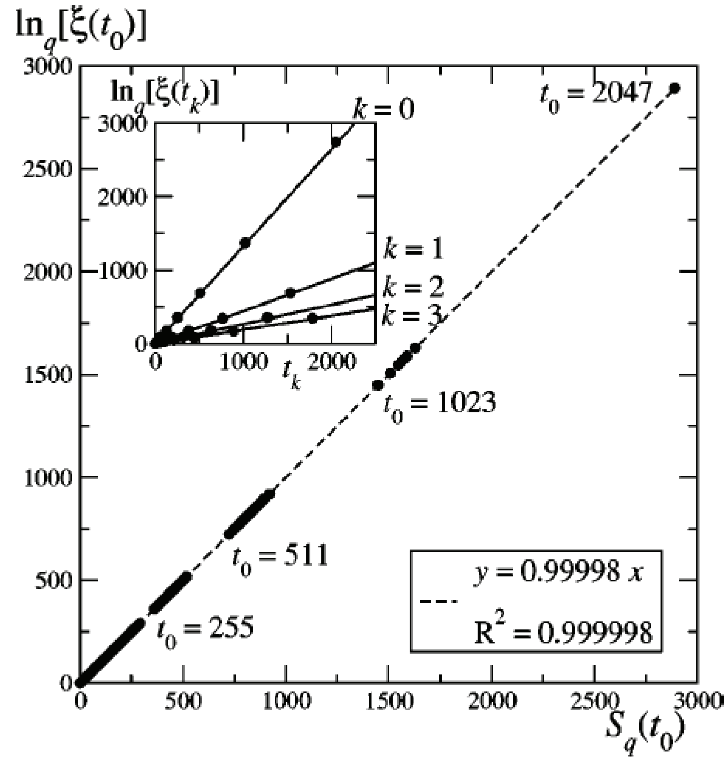
$$K_1 \equiv \lim_{t\to\infty} \frac{S_1(t)}{t}$$

and

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda_1\ t}$$

(weak chaos, i.e., **zero** Lyapunov exponent)





C. T. , A.R. Plastino and W.-M. Zheng, Chaos, Solitons & Fractals **8**, 885 (1997)
M.L. Lyra and C. T. , Phys Rev Lett **80**, 53 (1998)
V. Latora, M. Baranger, A. Rapisarda and C. T. , Phys Lett A **273**, 97 (2000)
E.P. Borges, C. T. , G.F.J. Ananos and P.M.C. Oliveira, Phys Rev Lett **89**, 254103 (2002)
F. Baldovin and A. Robledo, Phys Rev E **66**, R045104 (2002) and **69**, R045202 (2004)
G.F.J. Ananos and C. T. , Phys Rev Lett **93**, 020601 (2004)
E. Mayoral and A. Robledo, Phys Rev E **72**, 026209 (2005), and references therein

It can be proved that

$$K_q = \lambda_q \quad (q-\textit{generalized Pesin-like identity})$$

where

$$K_q \equiv \lim_{t\to\infty} \sup \left\{ \frac{S_q(t)}{t} \right\}$$

and

$$\xi(t) \equiv \sup \left\{ \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} \right\} = e_q^{\lambda_q \, t}$$

with

$$\frac{1}{1-q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}} = \frac{\ln \alpha_F}{\ln 2} \quad \textit{and} \quad \lambda_q = \frac{1}{1-q}$$

$$\left[x_{t+1} = 1 - a\,|x_t|^z \;\Rightarrow\; \frac{1}{1-q(z)} = \frac{1}{\alpha_{\min}(z)} - \frac{1}{\alpha_{\max}(z)} = (z-1)\frac{\ln \alpha_F(z)}{\ln 2} \right]$$

EDGE OF CHAOS OF THE LOGISTIC MAP:

(Using result in http://pi.lacim.uqam.ca/piDATA/feigenbaum.txt)

q =

0.2444877013412820661987704234046804052344469354900576736703650986327749672766558665755156226857540706288349640382728306063600193730331818964
5513410812778097921943860270831944900524658135215031745349520749404481
6546094908744833405672362246648808333307214231898714587299268154849677
4607864821834569063370205946820461899021675321457546117438305008496860
4088469694917043674789915060166464910602178348278899938183825225545823
3803811311803180544823675794499039707439546614634081555316878853503011
3821491411266246328940130370152354936571471269917921021622688833029675
4057806307068223688104320157903521237407354446029700060552504231420280
8919357881123973197797484423515245604092644670957957030465861412956647
9666687743683240492022757393004750895311855179558720483992696896827555
8524450244365268256094237801280330948779544035425248590433797618027118
3000457358555073894113675878440062913563042167454169409213569860320785
9088199859359007319336801069967496707904456092418632112054130547393985
795544410347612222592136846219346009360…

(1018 meaningful digits)

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Physics Letters A

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Sensitivity to initial conditions, entropy production, and escape rate at the onset of chaos

Miguel Angel Fuentes [a,b,c], Yuzuru Sato [d], Constantino Tsallis [a,e,*]

SENSITIVITY TO INITIAL CONDITIONS, ENTROPY AND ESCAPE RATE AT THE ONSET OF CHAOS

$$K_{q_{entropy\ production}} = \lambda_{q_{sensitivity}} - \gamma_{q_{escape}}$$

$$x_{t+1} = 1 - a\, x_t^z \qquad (z \geq 1;\ 0 \leq a \leq 2;\ -1 \leq x_t \leq 1)$$

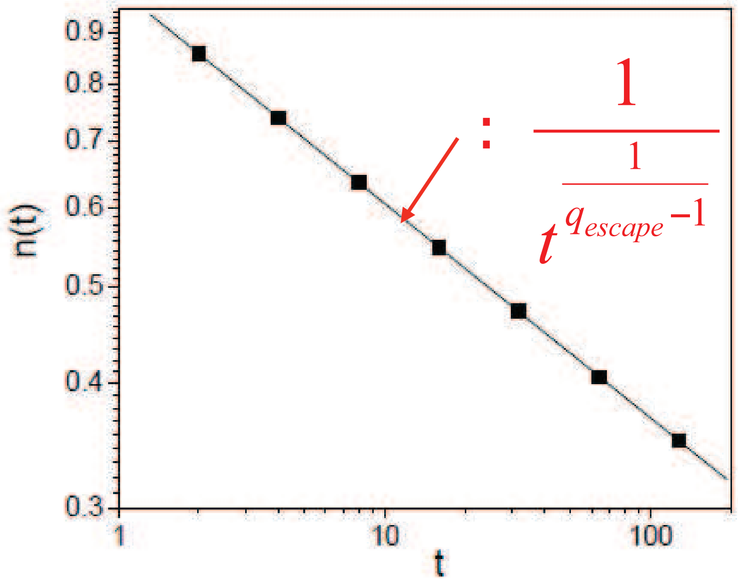


Figure 1. Fraction of points, $n(t) = N_0/N_t$, remaining in the system versus time using $\delta = 6/7 \sim 0.86$ and $z = 2$, in log-log scale. $N_0 = 10^6$ uniformly taken within the interval $[1 - 10^{-10}, 1]$. The fit, dashed line, shows a escape parameter $\gamma_{q_{esc}} = 0.216...$ while the theoretical one, calculated from Eq. (23), is $\gamma_{q_{esc}} = 0.2223....$

M.A. Fuentes, Y. Sato and C. T., Phys Lett A (2011), in press

M.A. Fuentes, Y. Sato and C. T., Phys Lett A (2011), in press

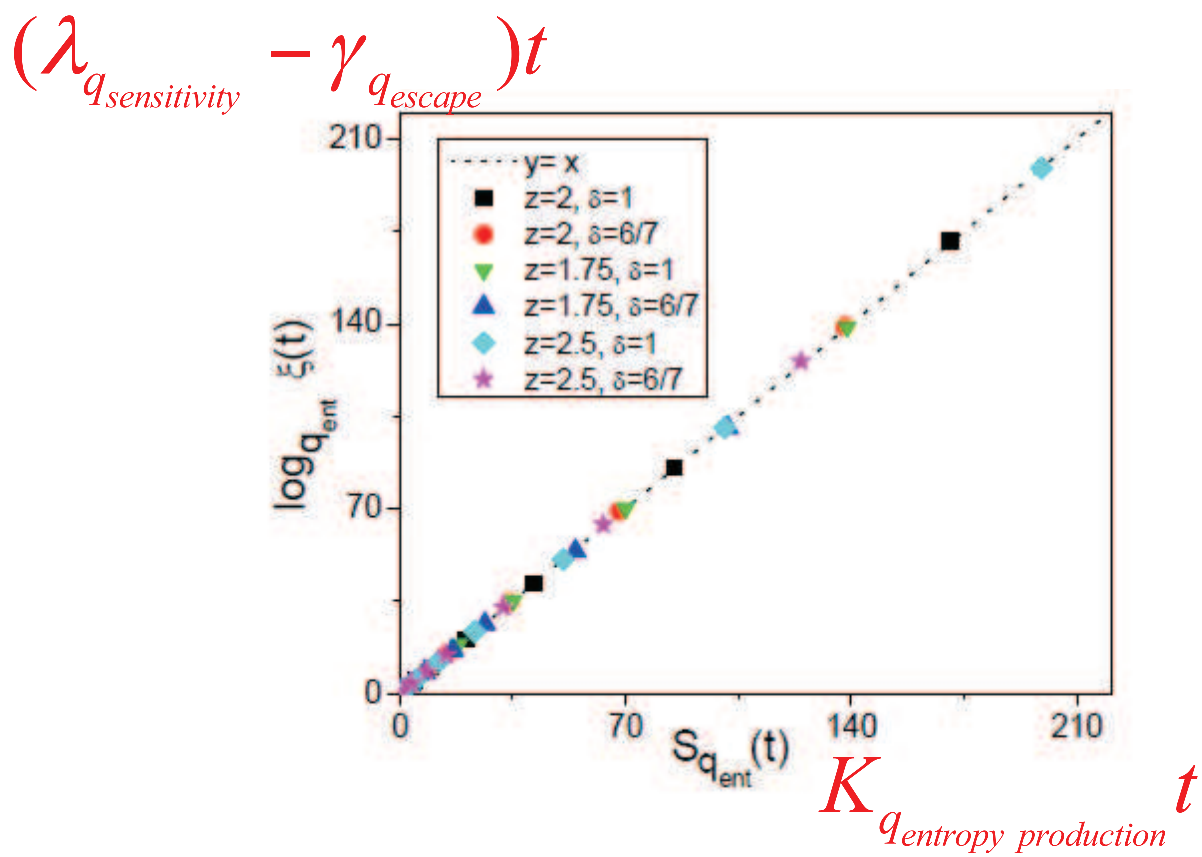
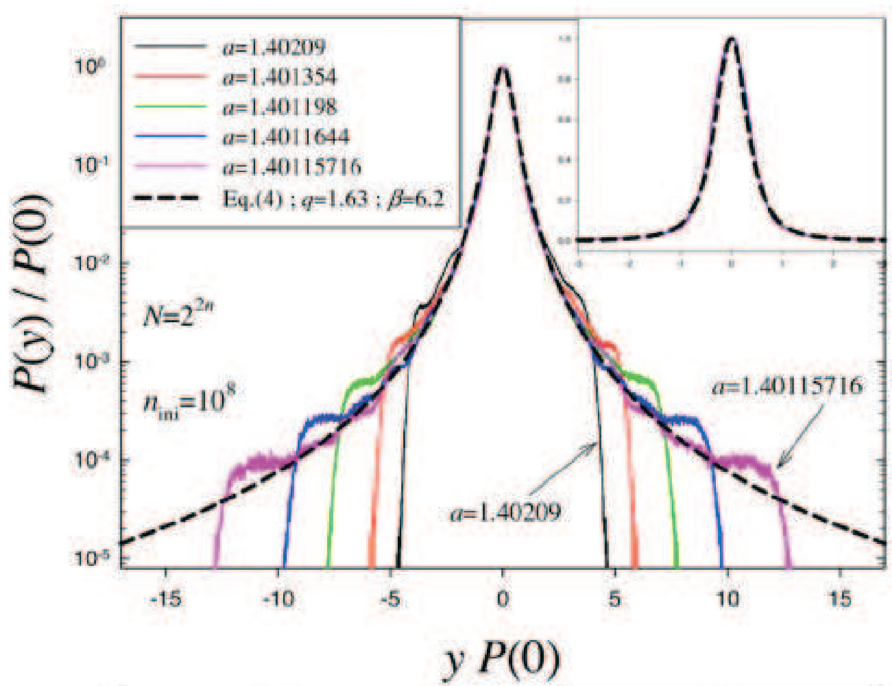


Figure 2. Sensitivity to initial condition versus entropy production, see Eqs. (16) and (17), for different values of z. For $z = 2$ and $\delta = 0$: $K_{q_{ent}} = \lambda_{q_{sen}} = 1.32...$, and $q_{ent} = q_{sen} = 0.244...$; while for $z = 2$ and $\delta = 6/7$: $\gamma_{q_{esc}} = 0.222...$, from Eq. (23), $K_{q_{ent}} = 1.1012...$ and $q_{ent} = 0.0919...$, from Eqs. (17) and (18). Similar results can be obtained for the other values of z. The holes are uniformly distributed in the line $y = 0$. The continuous line correspond to a fit with a slope 1.004..., numerically very close to unity, as expected. These examples neatly illustrate the validity of Eq. (17): the ordinate corresponds to $(\lambda_{q_{sen}} - \gamma_{q_{esc}})\ t$, and the abscissa corresponds to $K_{q_{ent}}\ t$.

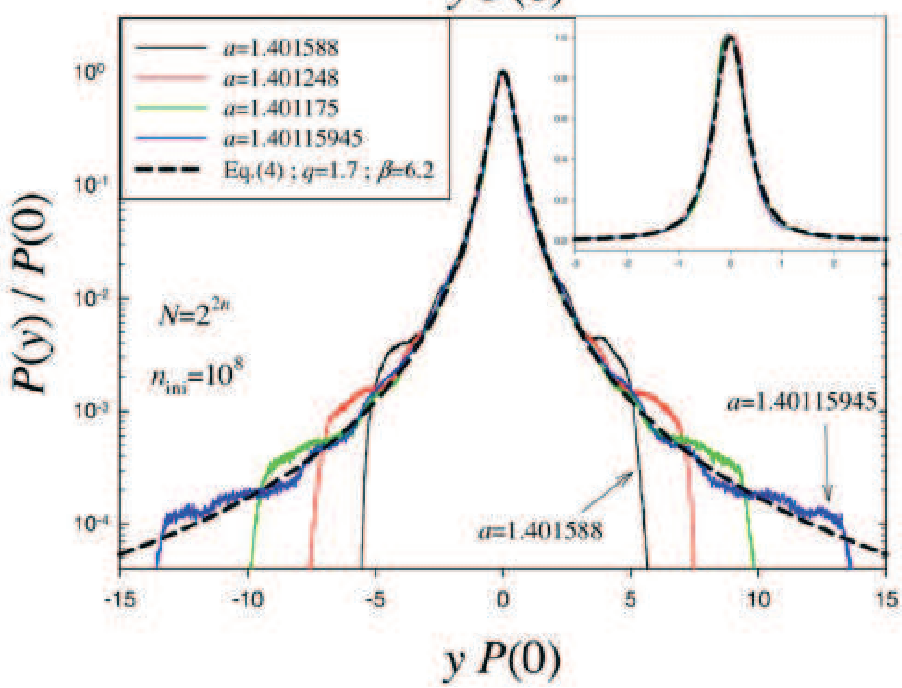
LOGISTIC MAP: EDGE OF CHAOS



odd 2n

q=1.63

beta=6.2



even 2n

q=1.70

beta=6.2

U. Tirnakli, C. Beck and C. T.
Phys Rev E **75**, 040106(R) (2007)

U. Tirnakli, C. T. and C. Beck
Phys Rev E **79**, 056209 (2009)

EDGE OF CHAOS OF THE LOGISTIC MAP:

$$q\text{-triplet}\begin{cases} q_{sensitivity} = q_{entropy} = 0.2444877013412820661 98\ldots \\ q_{relaxation} = 2.249784109\ldots \\ q_{stationary\ state} = 1.65 \pm 0.05 \end{cases}$$

In order to have

$$(q_{sensitivity}, q_{relaxation}, q_{stationary\ state}) \neq (1,1,1)$$

it seems that we need

$$maximal\ Lyapunov\ exponent = 0$$

What else do we need?

What relations may exist between these three q-indices?

CONSERVATIVE MC MILLAN MAP:

$$x_{n+1} = y_n$$

$$y_{n+1} = -x_n + 2\square \frac{y_n}{1 + y_n^2} + \varepsilon y_n$$

$\square \neq 0 \Leftrightarrow$ nonlinear dynamics

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2011), in press

$(\square,\varepsilon) = (1.6,1.2)$ $(\lambda_{max} \approx 0.05)$

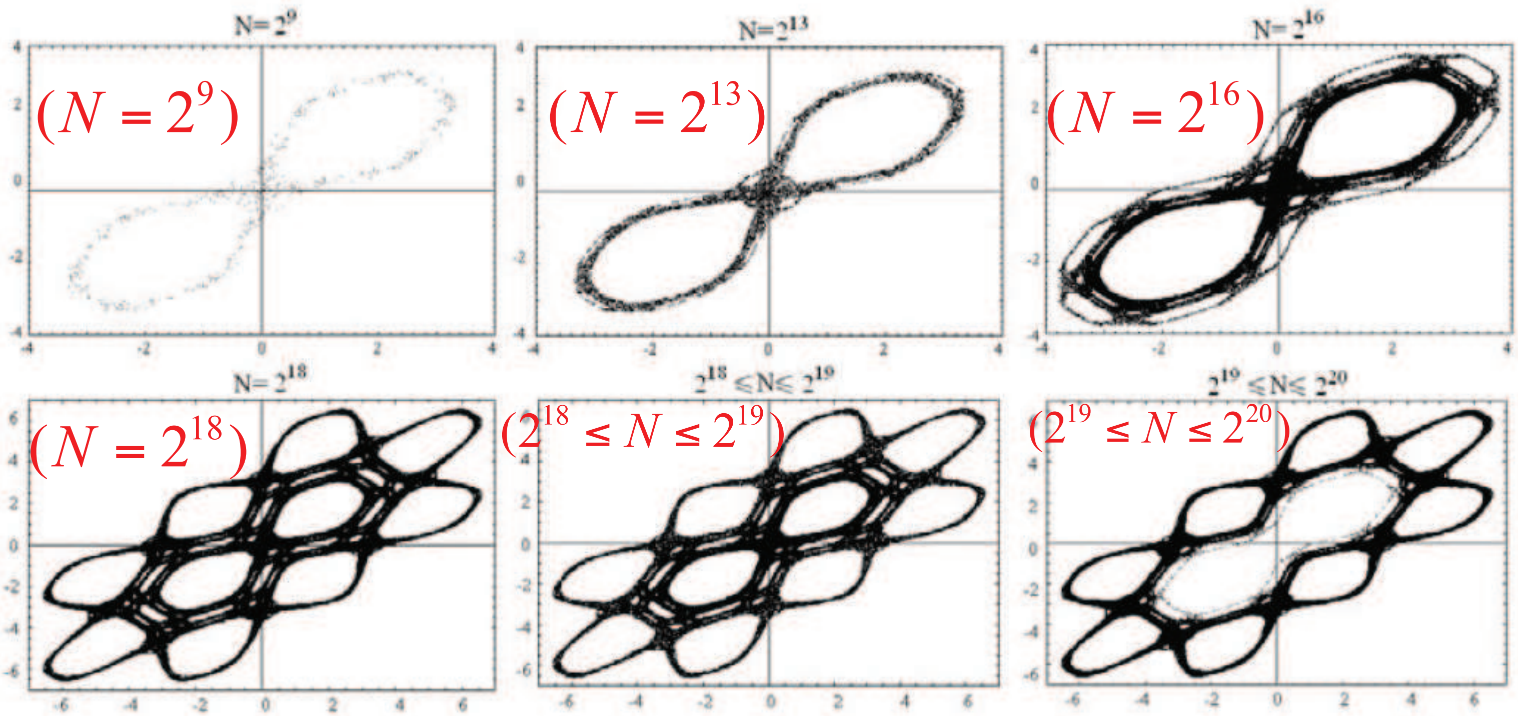
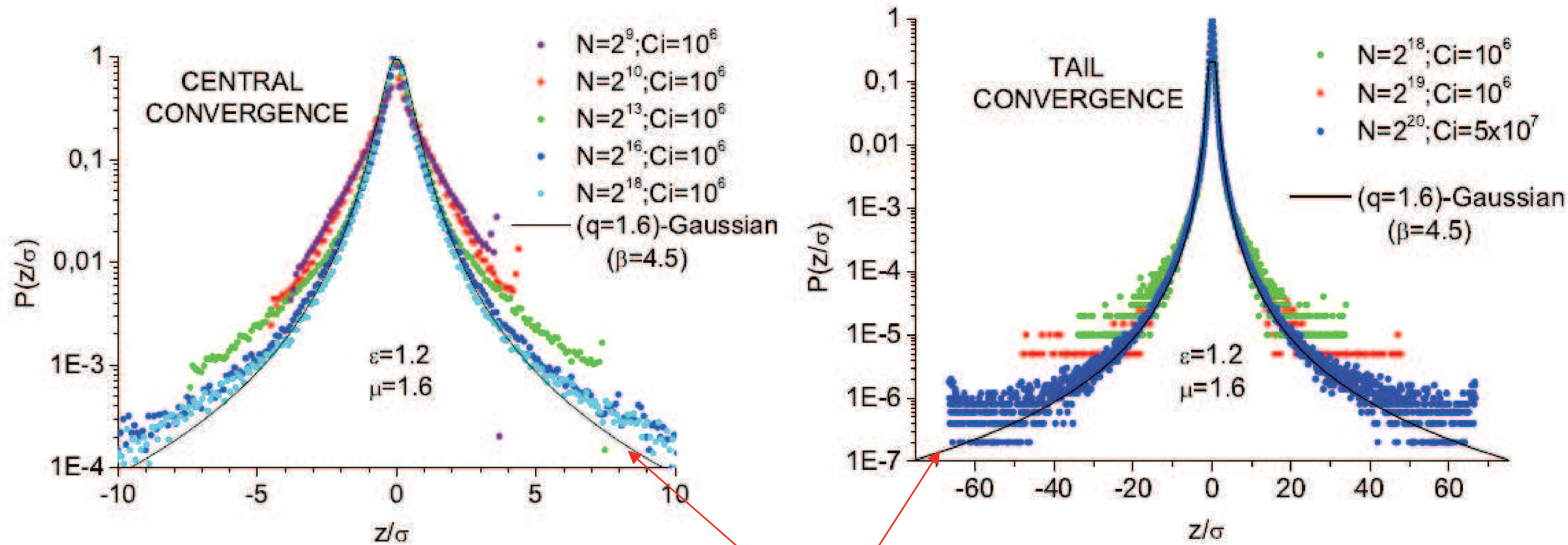


FIG. 10. Structure of phase space plot of Mc. Millan perturbed map for parameter values $\mu = 1.6$ and $\epsilon = 1.2$, starting form a randomly chosen initial condition in a square $(0, 10^{-6}) \times (0, 10^{-6})$, and for $i = 1 \dots N$ $(N = 2^{10}, 2^{13}, N^{16}, N^{18})$ iterates.

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2011), in press

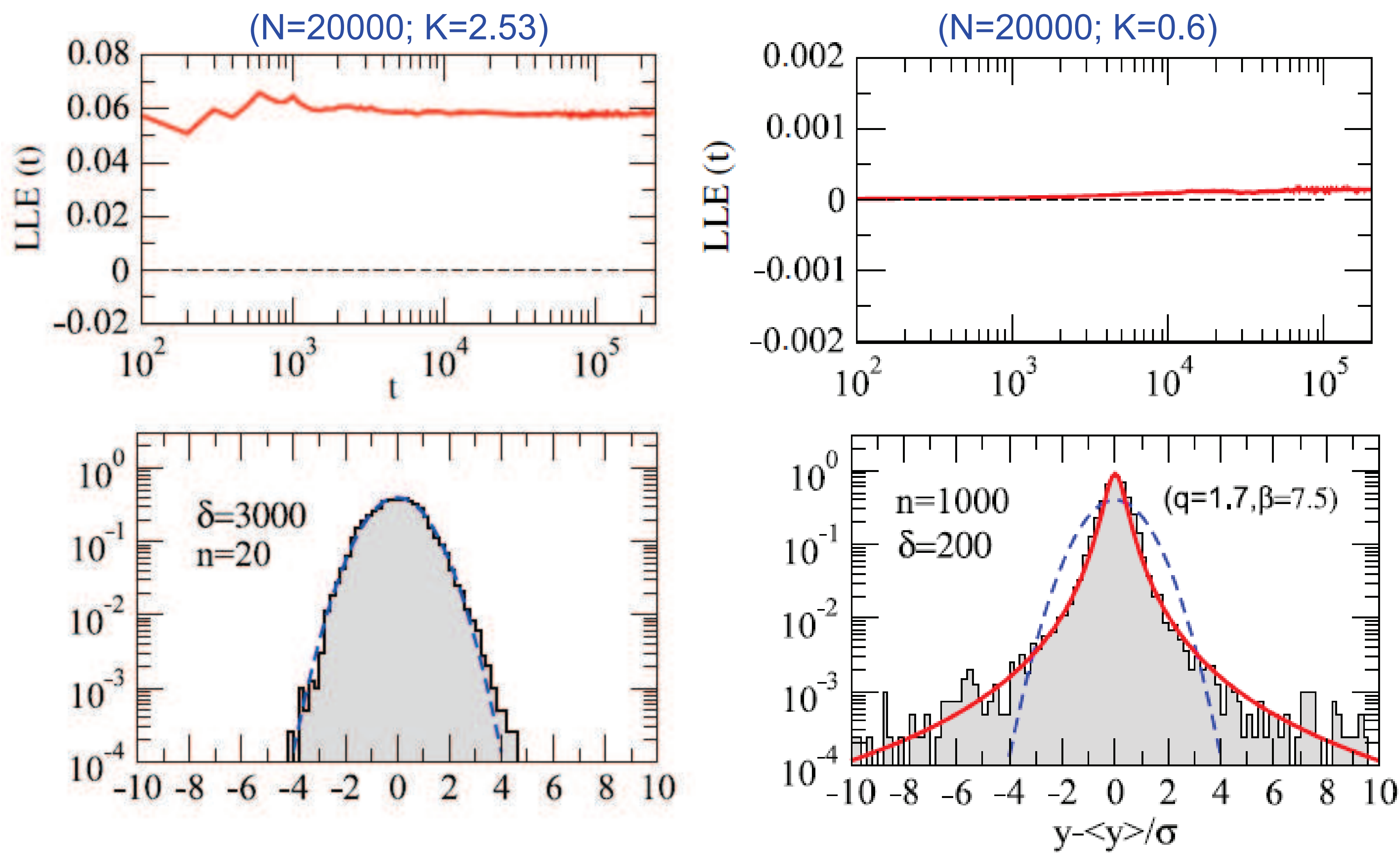


$$p \propto e_q^{-\beta (z/\sigma)^2}$$

with $(q, \beta) = (1.6, 4.5)$

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2011), in press

KURAMOTO MODEL: (N nonlinearly coupled oscillators)



G. Miritello, A. Pluchino and A. Rapisarda, Physica A **388**, 4818 (2009)

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Thermostatistics in the neighbourhood of the π-mode solution for the Fermi–Pasta–Ulam β system: from weak to strong chaos

Mario Leo[1], **Rosario Antonio Leo**[1] **and Piergiulio Tempesta**[2]

[1] Dipartimento di Fisica, Università del Salento, Via per Arnesano, 73100—Lecce, Italy
[2] Departamento de Física Teórica II, Facultad de Físicas, Ciudad Universitaria, Universidad Complutense, 28040—Madrid, Spain
E-mail: mario.leo@le.infn.it, leora@le.infn.it and p.tempesta@fis.ucm.es

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Online at stacks.iop.org/JSTAT/2010/P04021
doi:10.1088/1742-5468/2010/04/P04021

Abstract. We consider a π-mode solution of the Fermi–Pasta–Ulam β system. By perturbing it, we study the system as a function of the energy density from a regime where the solution is stable to a regime where it is unstable, first weakly and then strongly chaotic. We introduce, as an indicator of stochasticity, the ratio ρ (when it is defined) between the second and the first moment of a given probability distribution. We will show numerically that the transition between weak and strong chaos can be interpreted as the symmetry breaking of a set of suitable dynamical variables. Moreover, we show that in the region of weak chaos there is numerical evidence that the thermostatistic is governed by the Tsallis distribution.

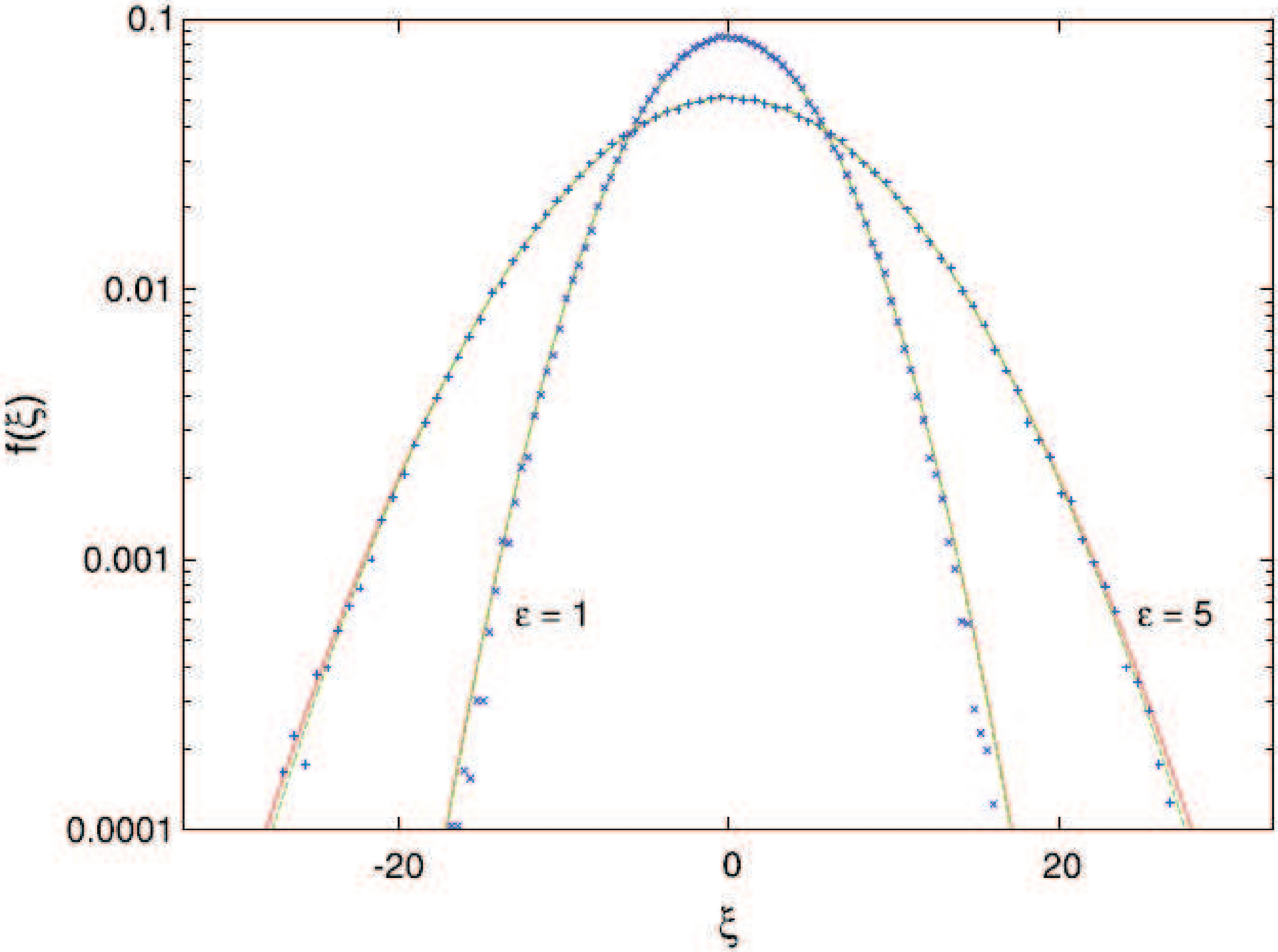


Figure 5. Plot on a linear-log scale of the numerical distribution $f(\xi)$ (blue points) fitted with a Tsallis distribution (red) and a Gauss distribution (green) for $N = 128$, $\epsilon = 1$ and 5. In both cases the Tsallis and Gaussian distributions essentially overlap.

M. Leo, R.A. Leo and P. Tempesta, J Stat Mech P04021 (2010)

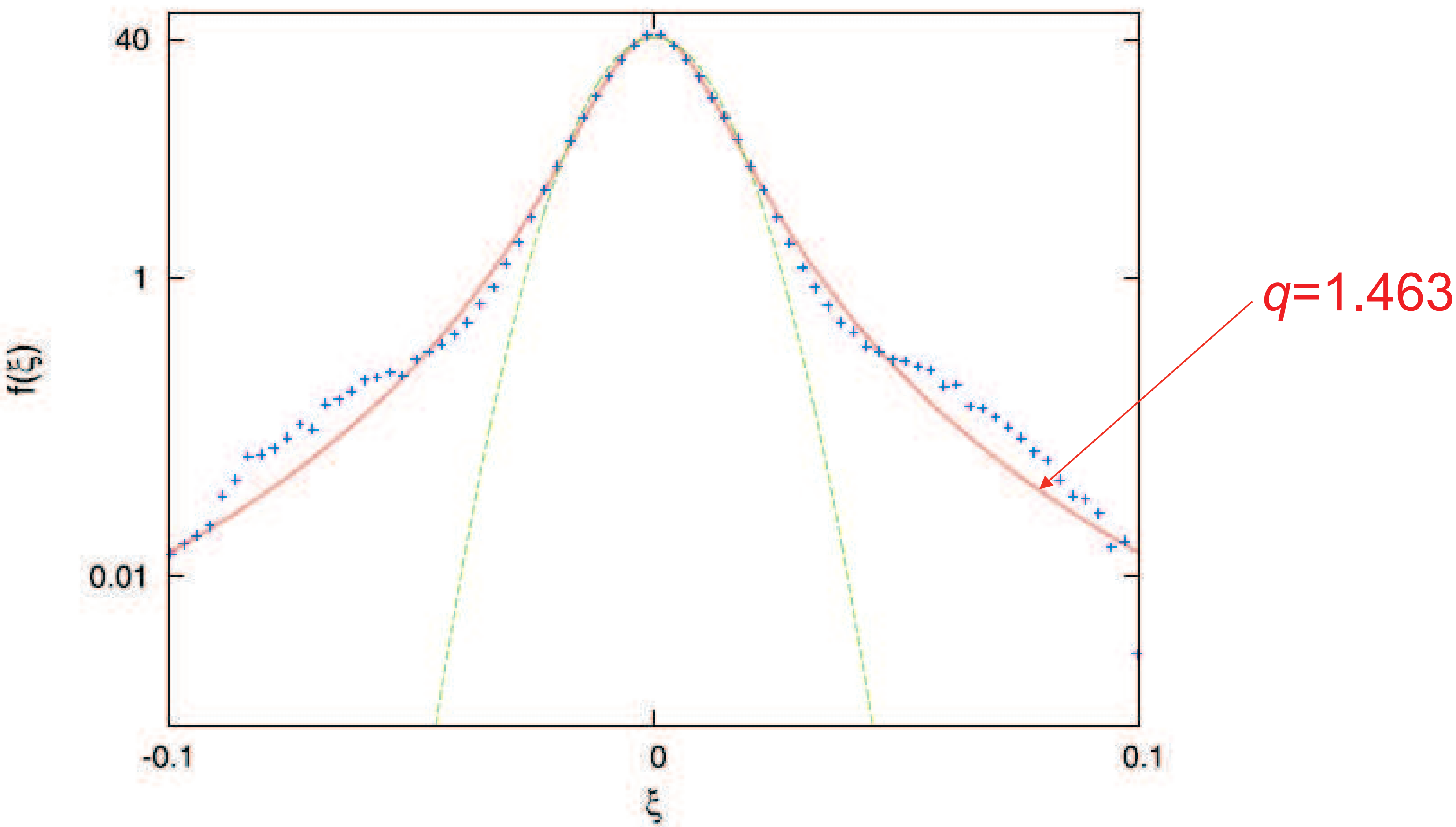


Figure 4. Plot on a linear-log scale of the numerical distribution $f(\xi)$ (blue points) fitted with a Tsallis distribution (red) and a Gauss distribution (green) for $N = 128$ and $\epsilon = 0.006$.

M. Leo, R.A. Leo and P. Tempesta, J Stat Mech P04021 (2010)

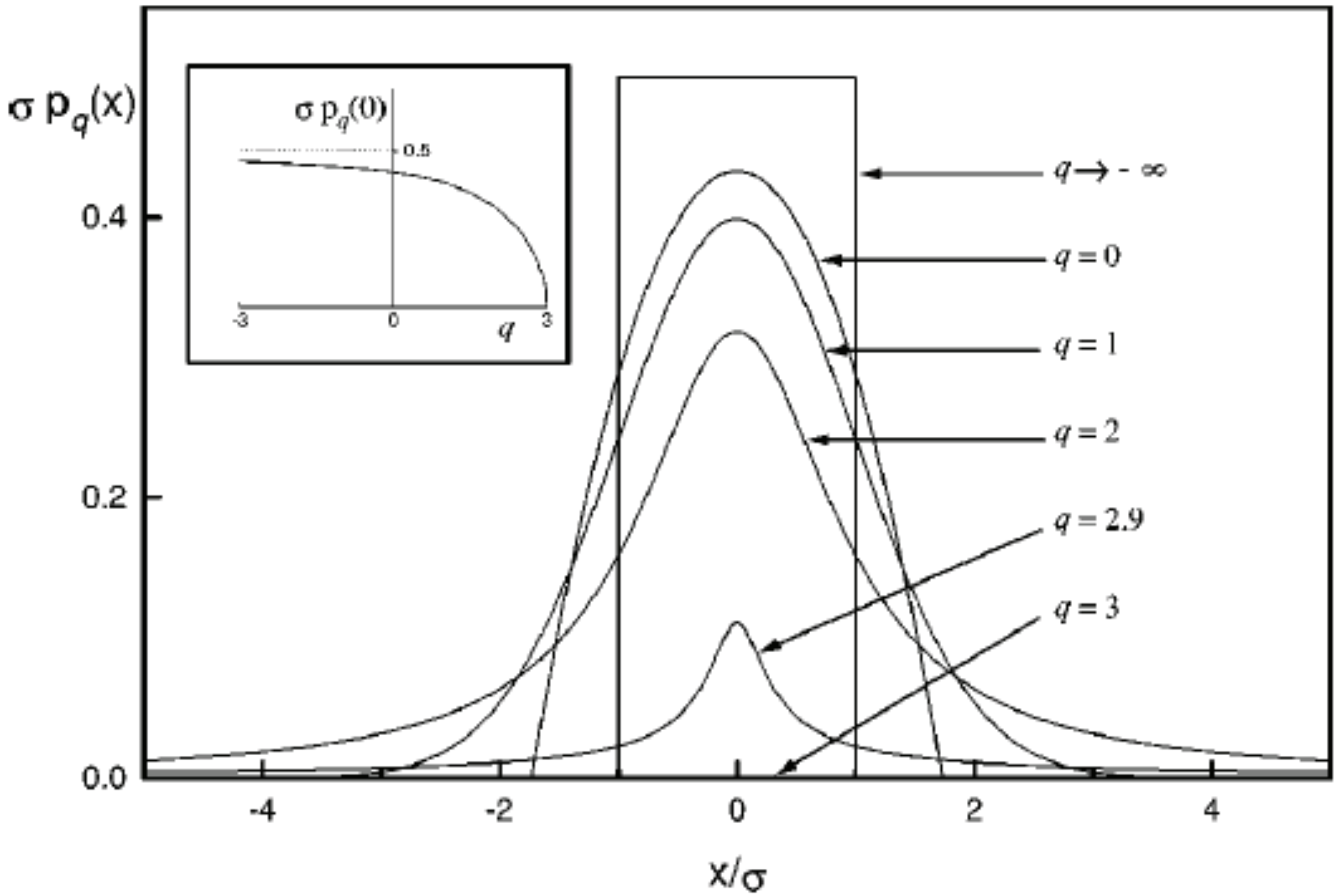
- Additive *versus* Extensive

- Nonlinear dynamical systems

- **Central Limit Theorem**

- Predictions, verifications, applications

q-GAUSSIANS: $p_q(x) \propto e_q^{-(x/\sigma)^2} \equiv \frac{1}{\left[1+(q-1)\,(x/\sigma)^2\right]^{\frac{1}{q-1}}}$ $(q<3)$



D. Prato and C. T., Phys Rev E **60**, 2398 (1999)

Milan j. math. 76 (2008), 307–328

1424-9286/010307-22, *published online* 14.3.2008
DOI 10.1007/s00032-008-0087-y

Milan Journal of Mathematics

On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

JOURNAL OF MATHEMATICAL PHYSICS **51**, 033502 (2010)

Generalization of symmetric α-stable Lévy distributions for $q>1$

Sabir Umarov,[1,a] Constantino Tsallis,[2,3,b] Murray Gell-Mann,[3,c] and Stanly Steinberg[4,d]
[1]*Department of Mathematics, Tufts University, Medford, Massachusetts 02155, USA*
[2]*Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*
[3]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*
[4]*Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, USA*

(Received 10 November 2009; accepted 4 January 2010; published online 3 March 2010)

q – PRODUCT:

L. Nivanen, A. Le Mehaute and Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003)
E.P. Borges, Physica A **340**, 95 (2004)

The q - product is defined as follows:

$$x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}$$

$$Properties:$$

$$i) \quad x \otimes_1 y = x\, y$$

$$ii) \quad \ln_q(x \otimes_q y) = \ln_q x + \ln_q y$$ (extensivity of Sq)

$$[whereas \;\; \ln_q(x\, y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y)]$$

(nonadditivity of Sq)

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\, dx = \int_{-\infty}^{\infty} e_q^{ix\xi[f(x)]^{q-1}} f(x)\, dx$$

$(q \geq 1)$

(nonlinear!)

For q<1 see K.P. Nelson and S. Umarov, Physica A **389**, 2157 (2010)

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-independence:

Two random variables X [*with density* $f_X(x)$] *and* Y [*with density* $f_Y(y)$] *having zero* $q-$*mean values are said* *q - independent* *if*

$$F_q[\mathrm{X+Y}](\xi) = F_q[\mathrm{X}](\xi) \otimes_{\frac{1+q}{3-q}} F_q[\mathrm{Y}](\xi) \quad ,$$

i.e., if

$$\int_{-\infty}^{\infty} dz\; e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[\int_{-\infty}^{\infty} dx\; e_q^{ix\xi} \otimes_q f_X(x)\right] \otimes_{(1+q)/(3-q)} \left[\int_{-\infty}^{\infty} dy\; e_q^{iy\xi} \otimes_q f_Y(y)\right] ,$$

with

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; h(x,y)\; \delta(x+y-z) = \int_{-\infty}^{\infty} dx\; h(x, z-x) = \int_{-\infty}^{\infty} dy\; h(z-y, y)$$

where $h(x,y)$ *is the joint density.*

$$q\text{-}independence\ means \begin{cases} independence & if\ \ q = 1\ ,\ i.e.,\ h(x,y) = f_X(x)\, f_Y(y) \\ global\ correlation & if\ \ q \neq 1\ ,\ i.e.,\ h(x,y) \neq f_X(x)\, f_Y(y) \end{cases}$$

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$ *-scaled attractor* $\mathrm{F}(x)$ *when summing* $N \to \infty$ q*-independent identical random variables*

with symmetric distribution $f(x)$ *with* $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q \equiv 2q-1,\ q_1 = \dfrac{1+q}{3-q} \right)$

	$q=1$ [*independent*]	$q \neq 1$ (*i.e.*, $Q \equiv 2q-1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathrm{F}(x) = \mathit{Gaussian}\ G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathrm{F}(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } \vert x \vert \ll x_c(q,2) \\ f(x) \sim C_q / \vert x \vert^{2/(q-1)} & \text{if } \vert x \vert \gg x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathrm{F}(x) = \mathit{Levy\ distribution}\ L_\alpha(x)$, *with same* $\vert x \vert \to \infty$ *behavior* $\mathrm{L}_\alpha(x) \sim \begin{cases} G(x) \\ \quad \text{if } \vert x \vert \ll x_c(1,\alpha) \\ f(x) \sim C_\alpha / \vert x \vert^{1+\alpha} \\ \quad \text{if } \vert x \vert \gg x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathrm{F}(x) = L_{q,\alpha}$, *with same* $\vert x \vert \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\, \alpha}(x) \sim C^*_{q,\alpha} / \vert x \vert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \quad (\mathit{intermediate\ regime}) \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\, 2}(x) \sim C^L_{q,\alpha} / \vert x \vert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \quad (\mathit{distant\ regime}) \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg J Math Phys 51, 033502 (2010)

Hilhorst function:

[H.J. Hilhorst, JSTAT P 10023 (2010)]

$$f_A(x) = \begin{cases} \dfrac{\left[\;|x|^{(q-2)/(q-1)} - A\right]^{1/(q-2)}}{C_q\;|x|^{1/(q-1)}\;\left\{1+(q-1)\left[\;|x|^{(q-2)/(q-1)} - A\right]^{2(q-1)/(q-2)}\right\}^{1/(q-1)}} \\ \qquad\qquad\qquad\qquad \text{if } \; 0 \le A < |x|^{(q-2)/(q-1)} \\ 0 \qquad\qquad\qquad\qquad \text{if } \; 0 \le |x|^{(q-2)/(q-1)} \le A \\ \qquad \text{with} \quad \int_{-\infty}^{\infty} dx\, f_A(x) = 1 \end{cases}$$

Particular case: $A = 0$

$$f_0(x) = \frac{1}{C_q\;[1+(q-1)x^2]^{1/(q-1)}}$$

Physics Letters A 375 (2011) 2085–2088

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q-Generalization of the inverse Fourier transform

M. Jauregui [a,*], C. Tsallis [a,b]

[a] Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
[b] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

q-GENERALIZED INVERSE FOURIER TRANSFORM:

$$f(y) = \left[\frac{2-q}{2\pi} \int_{-\infty}^{+\infty} F_q[f(x+y)](\xi, y)\, d\xi \right]^{1/(2-q)} \qquad (1 \le q < 2)$$

Particular case *q* = 1:

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F[f(x+y)](\xi, y)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F[f(x)](\xi)\, e^{-i\xi y}\, d\xi$$

M. Jauregui and C. T., Phys Lett A **375**, 2085 (2011)

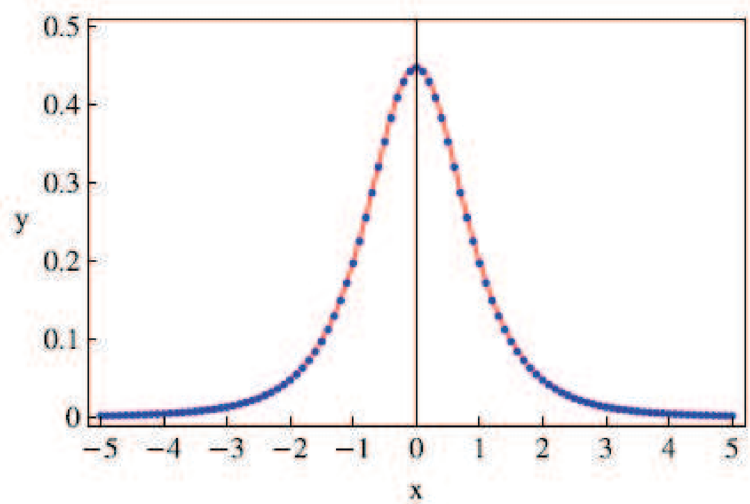


Fig. 1. Representation of $G_{3/2,1}(x)$. The continuous line corresponds to the analytical expression of the function; the dots were obtained by handling numerically Eqs. (1) and (6).

q **- Gaussian**

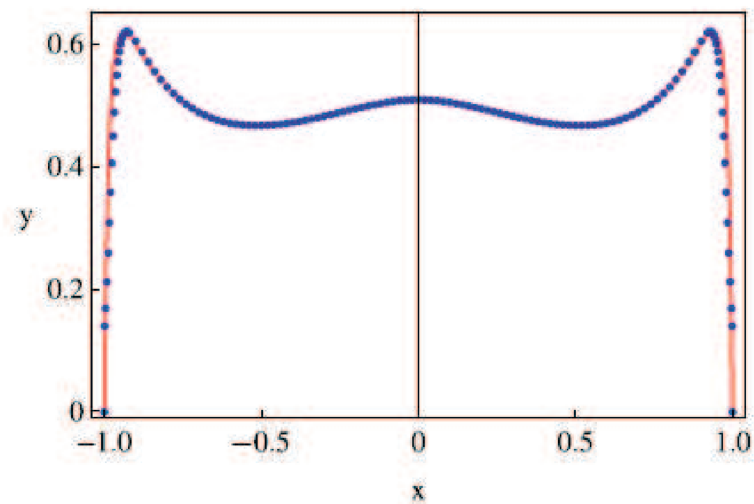


Fig. 2. Representation of $f_{1,5/4}(x)$. The continuous line corresponds to the analytical expression of the function; the dots were obtained by handling numerically Eqs. (1) and (6). For all values of $x \in (-1, 1)$ we have used $\gamma = 2$ in Eq. (6), whereas for $x = \pm 1$ we have used $\gamma = 1$.

Hilhorst function

M. Jauregui and C. T.
Phys Lett A **375**, 2085 (2011)

WHAT IS THE PHYSICAL MEANING OF *q*-INDEPENDENCE?

IS IT CONSISTENT WITH (STRICT OR ASYMPTOTIC) SCALE INVARIANCE? IF YES, IS IT SUFFICIENT? NECESSARY?

CANDIDATE MODELS FOR *q*-INDEPENDENCE:

1) *N* compact-support continuous variables with correlation introduced through a *N*-variate covariance matrix **(strictly scale-invariant)**

W. Thistleton, J.A. Marsh, K. Nelson and C. T., Cent. Eur. J. Phys. **7**, 387 (2009)

(see H.J. Hilhorst and G. Schehr, J Stat Mech (2007) P06003)

2) *N* binary variables with correlation introduced through the *q*-product **(strictly scale-invariant)**

L.G. Moyano, C. T. and M. Gell-Mann, Europhys Lett **73** (2006) 813

(see H.J. Hilhorst and G. Schehr, J Stat Mech (2007) P06003)

3) *N* binary variables with correlation introduced through a family of triangles generalizing the Leibnitz one **(strictly scale-invariant)**

A. Rodriguez, V. Schwammle and C. T., J Stat Mech (2008) P09006

R. Hanel, S. Thurner and C. T., Eur Phys J B **72**, 263 (2009)

4) N-binary-discretized *q*-Gaussians **(asymptotically scale-invariant)**

A. Rodriguez, V. Schwammle and C. T., J Stat Mech (2008) P09006

q-independence $\Rightarrow$ scale-invariance ?

i.e.,

$$\int dx_N h_N(x_1, x_2, \ldots, x_N) = h_{N-1}(x_1, x_2, \ldots, x_{N-1}) \quad ?$$

- Additive *versus* Extensive

- Nonlinear dynamical systems

- Central Limit Theorem

- **Predictions, verifications, applications**

COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:

RAPID COMMU

PHYSICAL REVIEW A **67**, 051402(R) (2003)

Anomalous diffusion and Tsallis statistics in an optical lattice

Eric Lutz
Sloane Physics Laboratory, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120
(Received 26 February 2003; published 27 May 2003)

We point out a connection between anomalous transport in an optical lattice and Tsallis' generalized statistics. Specifically, we show that the momentum equation for the semiclassical Wigner function which describes atomic motion in the optical potential, belongs to a class of transport equations recently studied by Borland [Phys. Lett. A **245**, 67 (1998)]. The important property of these ordinary linear Fokker-Planck equations is that their stationary solutions are exactly given by Tsallis distributions. An analytical expression of the Tsallis index q in terms of the microscopic parameters of the quantum-optical problem is given and the spatial coherence of the atomic wave packets is discussed.

(i) The distribution of atomic velocities is a *q*-Gaussian;

(ii) $q = 1 + \dfrac{44 E_R}{U_0}$ where $E_R \equiv$ recoil energy

$U_0 \equiv$ potential depth

Tunable Tsallis Distributions in Dissipative Optical Lattices

P. Douglas, S. Bergamini, and F. Renzoni

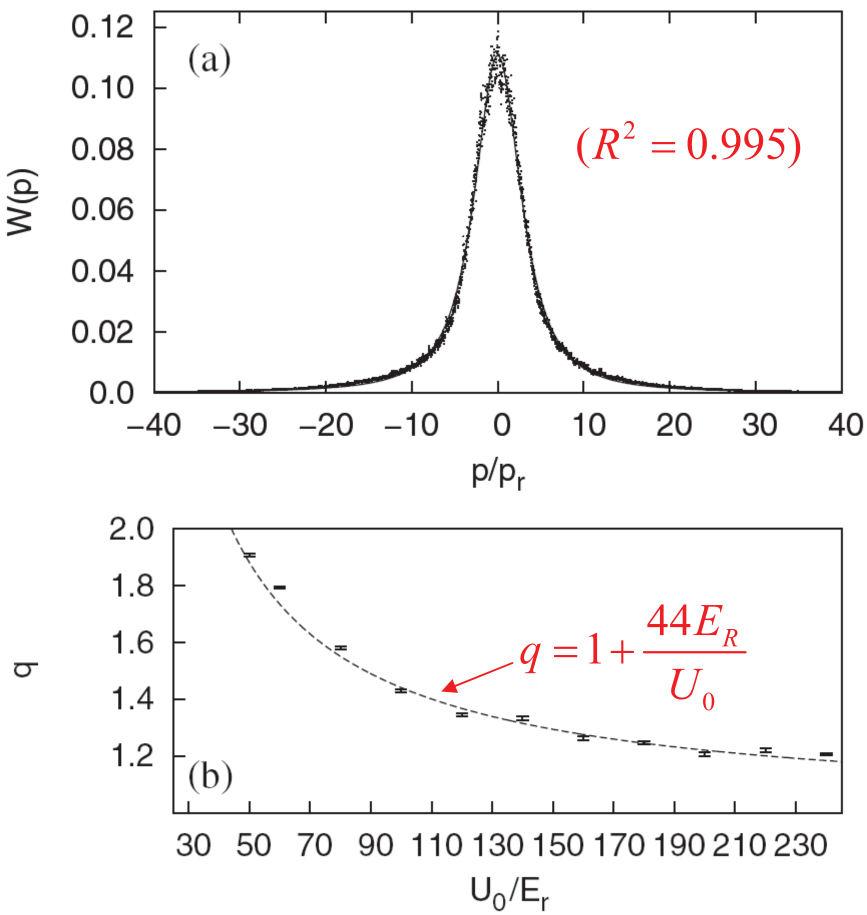
Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom

(Received 10 January 2006; published 24 March 2006)

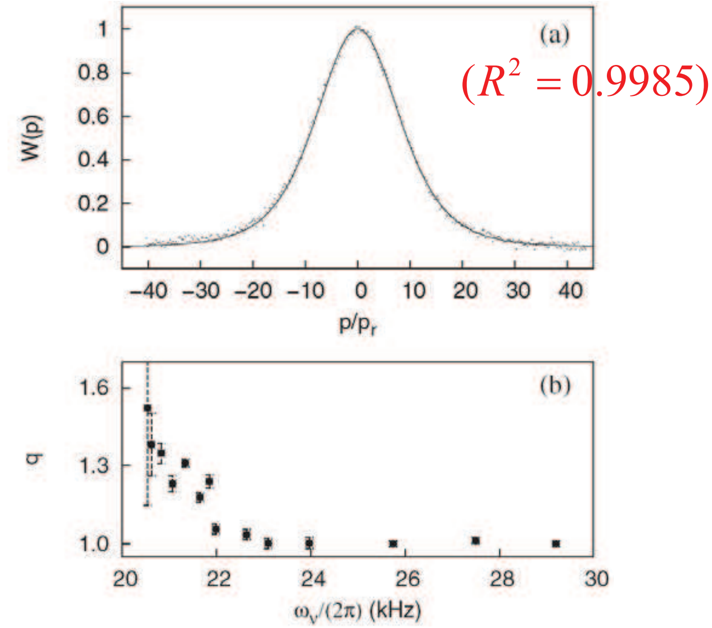
We demonstrated experimentally that the momentum distribution of cold atoms in dissipative optical lattices is a Tsallis distribution. The parameters of the distribution can be continuously varied by changing the parameters of the optical potential. In particular, by changing the depth of the optical lattice, it is possible to change the momentum distribution from Gaussian, at deep potentials, to a power-law tail distribution at shallow optical potentials.

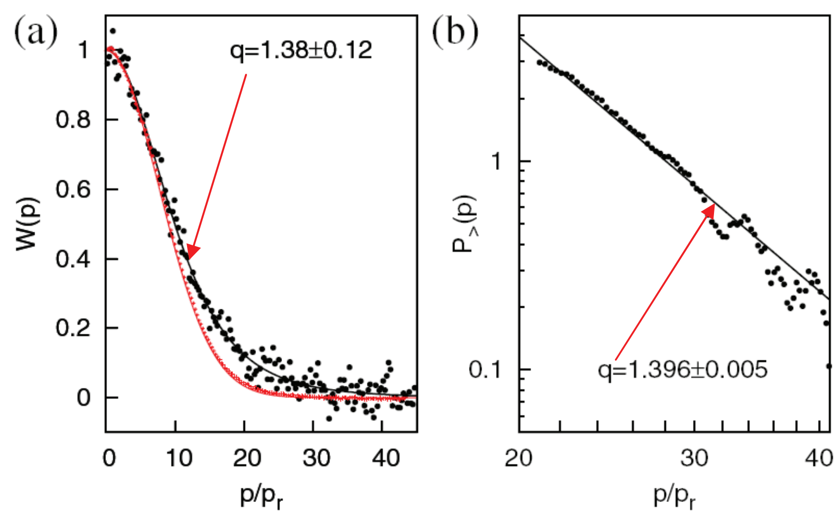
Experimental and computational verifications

by P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)



(Computational verification:
quantum Monte Carlo simulations)





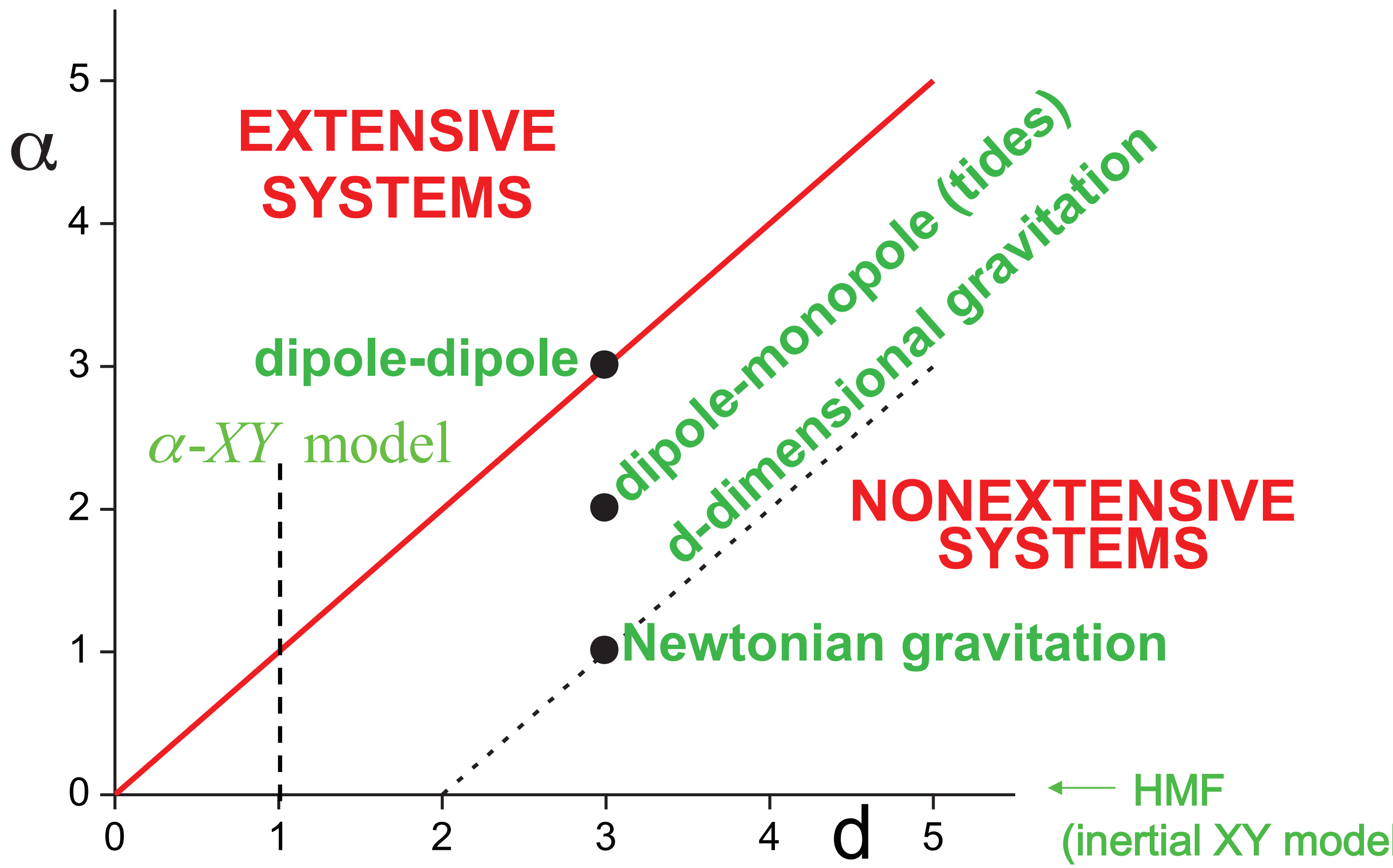
(Experimental verification: Cs atoms)

CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

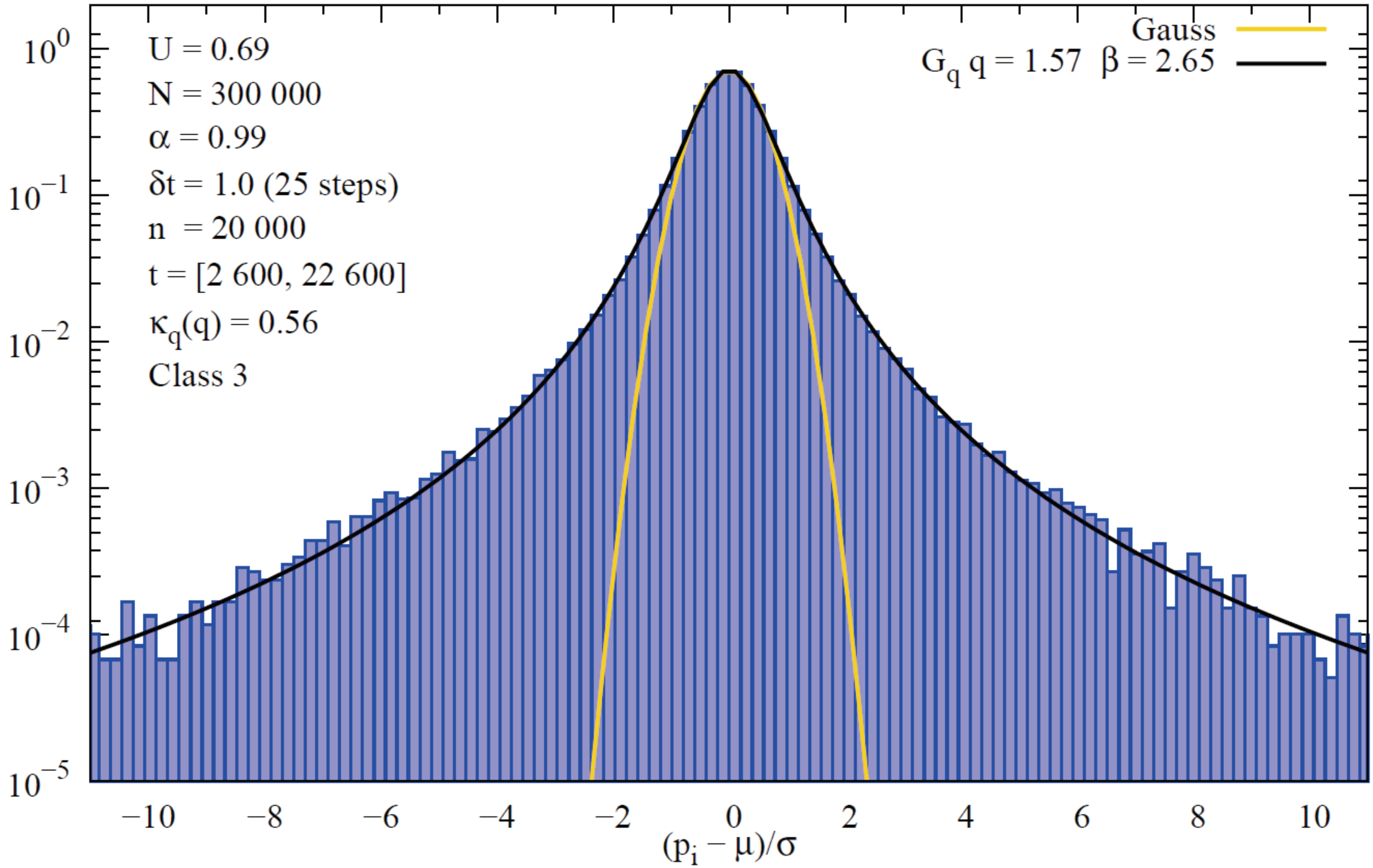
$$V(\vec{r}) \sim -\frac{A}{r^{\alpha}} \qquad (r \to \infty) \qquad (A > 0, \;\; \alpha \geq 0)$$

$$\textit{integrable if} \quad \alpha / d > 1 \qquad (\textit{short-ranged})$$

$$\textit{non-integrable if} \quad 0 \leq \alpha / d \leq 1 \qquad (\textit{long-ranged})$$



$d = 1 \quad \alpha - \mathrm{XY}$ model



L.J.L. Cirto, V.R.V. Assis and C. T. (2011)

PHYSICAL REVIEW E **00**, 001100 (2011)

Group entropies, correlation laws, and zeta functions

Piergiulio Tempesta*

Departamento de Física Teórica II, Facultad de Físicas, Ciudad Universitaria, Universidad Complutense, E-28040 Madrid, Spain

(Received 15 February 2011; revised manuscript received 3 May 2011; published xxxxx)

The notion of group entropy is proposed. It enables the unification and generaliztion of many different definitions of entropy known in the literature, such as those of Boltzmann-Gibbs, Tsallis, Abe, and Kaniadakis. Other entropic functionals are presented, related to nontrivial correlation laws characterizing universality classes of systems out of equilibrium when the dynamics is weakly chaotic. The associated thermostatistics are discussed. The mathematical structure underlying our construction is that of formal group theory, which provides the general structure of the correlations among particles and dictates the associated entropic functionals. As an example of application, the role of group entropies in information theory is illustrated and generalizations of the Kullback-Leibler divergence are proposed. A new connection between statistical mechanics and zeta functions is established. In particular, Tsallis entropy is related to the classical Riemann zeta function.

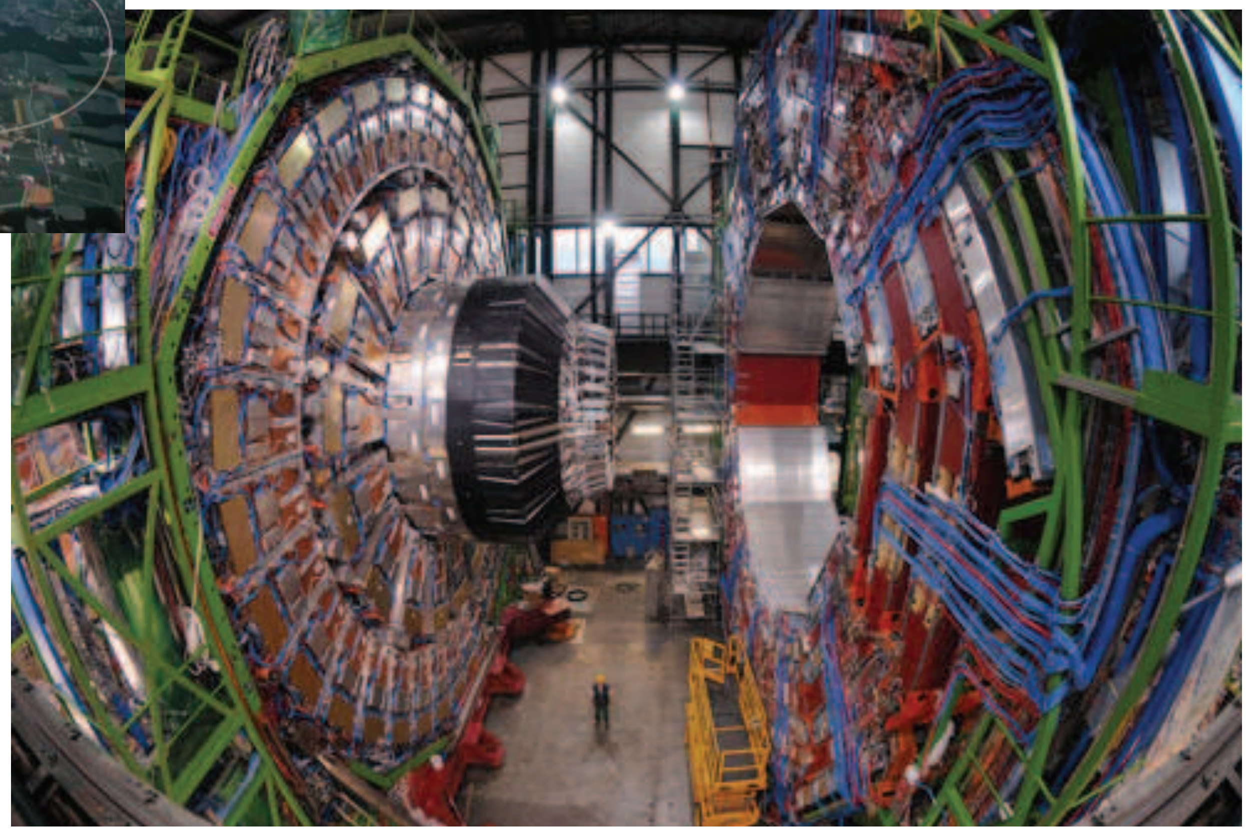
$$S_q \leftrightarrow \varsigma(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

$$= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdot \frac{1}{1-11^{-s}} \cdots$$

LHC (Large Hadron Collider)

CMS (Compact Muon Solenoid) detector

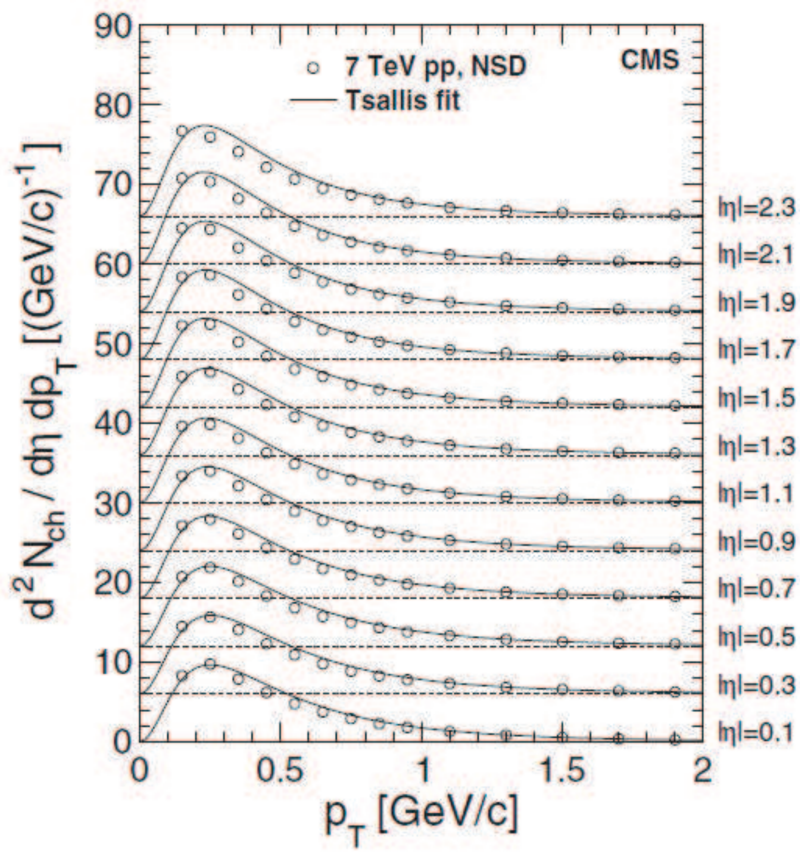
~ 2500 scientists/engineers from 183 institutions of 38 countries

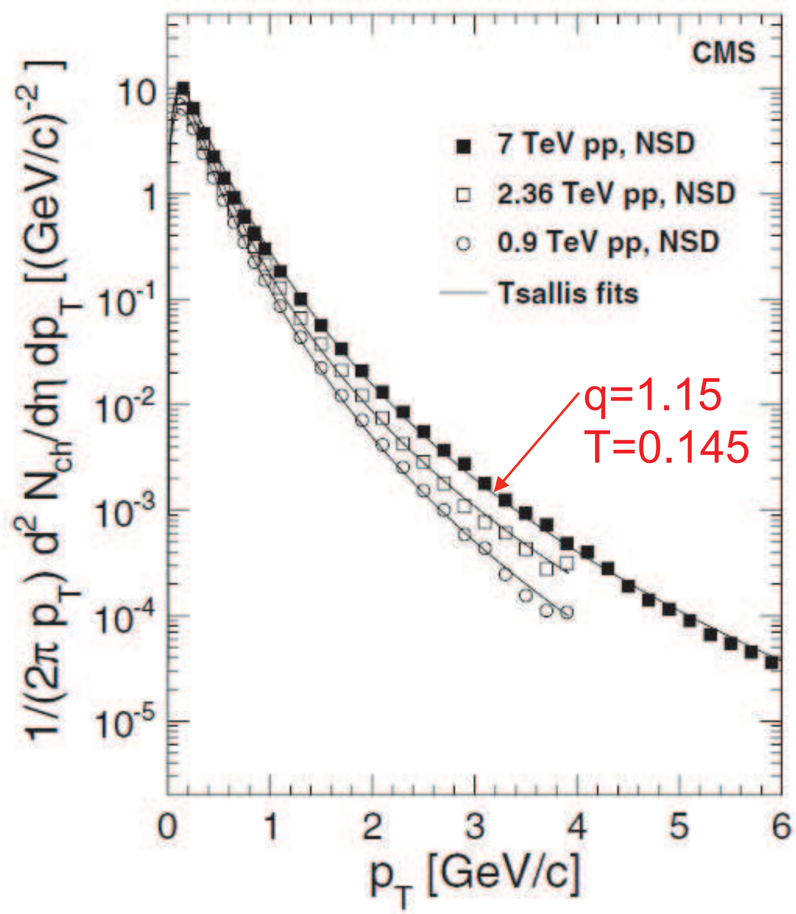


Transverse-Momentum and Pseudorapidity Distributions of Charged Hadrons in pp Collisions at $\sqrt{s} = 7$ TeV

V. Khachatryan *et al.**
(CMS Collaboration)

(Received 18 May 2010; published 6 July 2010)





PHYSICAL REVIEW D **83**, 052004 (2011)

Measurement of neutral mesons in $p + p$ collisions at $\sqrt{s} = 200$ GeV and scaling properties of hadron production

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PHENIX @ RHIC

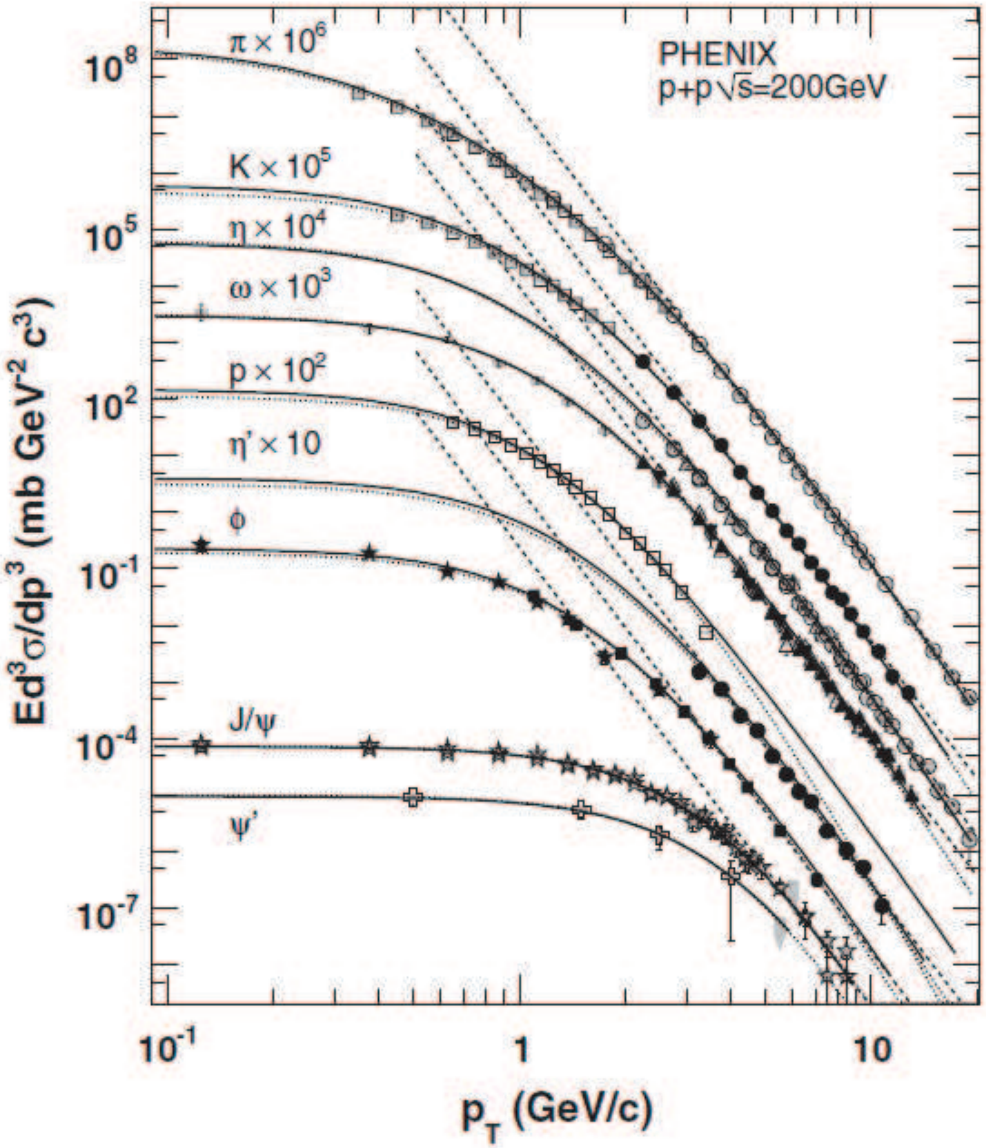

FIG. 13. The p_T spectra of various hadrons measured by PHENIX fitted to the power law fit (dashed lines) and Tsallis fit (solid lines). See text for more details.

$q \approx 1.10$

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A novel automatic microcalcification detection technique using Tsallis entropy & a type II fuzzy index

Mohanalin *, Beenamol, Prem Kumar Kalra, Nirmal Kumar

Department of Electrical Engineering, IIT Kanpur, UP-208016, India

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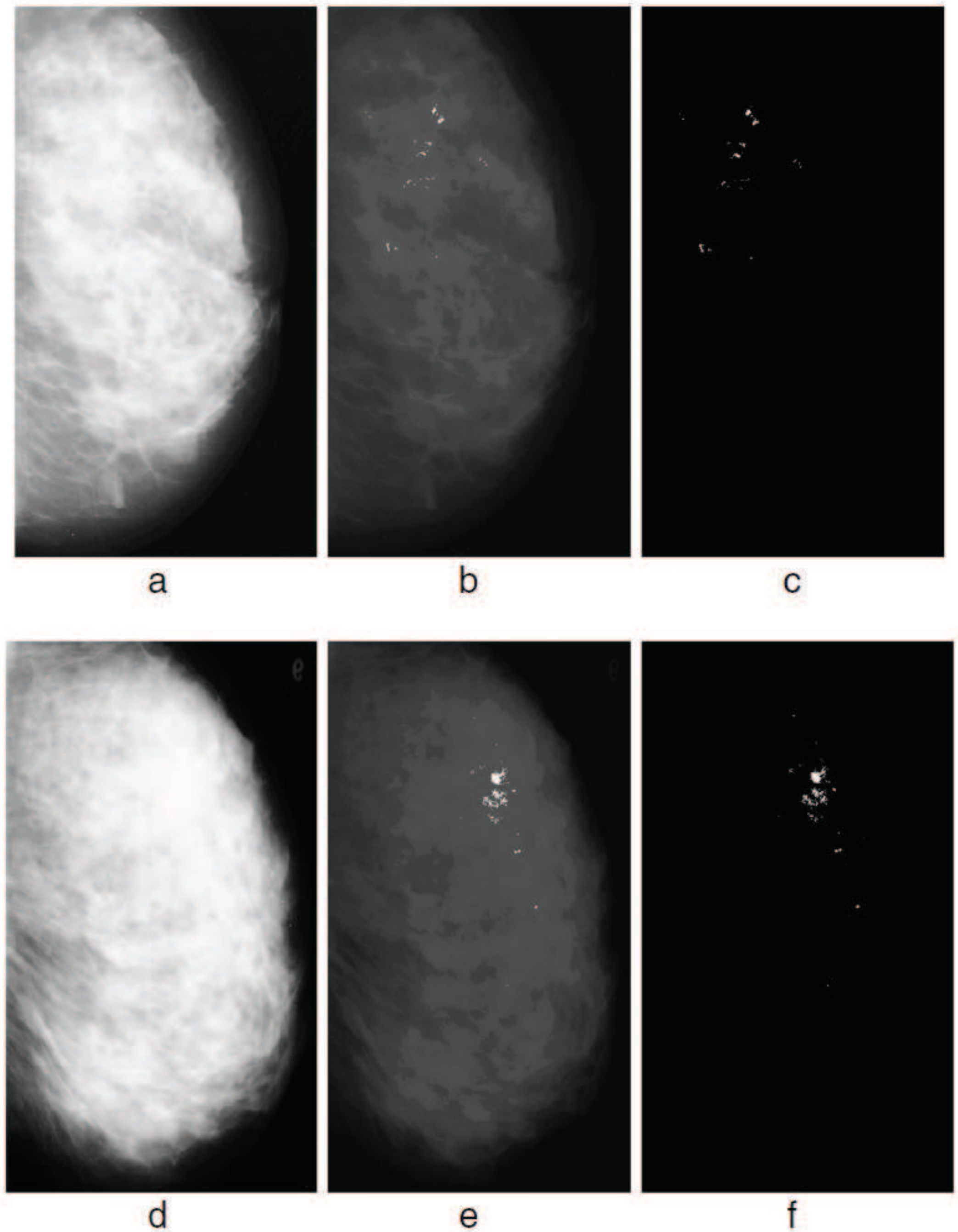
This article investigates a novel automatic microcalcification detection method using a type II fuzzy index. The thresholding is performed using the Tsallis entropy characterized by another parameter 'q', which depends on the non-extensiveness of a mammogram. In previous studies, 'q' was calculated using the histogram distribution, which can lead to erroneous results when pectoral muscles are included. In this study, we have used a type II fuzzy index to find the optimal value of 'q'. The proposed approach has been tested on several mammograms. The results suggest that the proposed Tsallis entropy approach outperforms the two-dimensional non-fuzzy approach and the conventional Shannon entropy partition approach. Moreover, our thresholding technique is completely automatic, unlike the methods of previous related works. Without Tsallis entropy enhancement, detection of microcalcifications is meager: 80.21% Tps (true positives) with 8.1 Fps (false positives), whereas upon introduction of the Tsallis entropy, the results surge to 96.55% Tps with 0.4 Fps.


a
b
c
d
e
f

q-PLANE WAVES:

1) New representation of Dirac delta:

$$\delta(x) = \frac{2-q}{2\pi}\int_{-\infty}^{\infty} dk\ e_q^{-ikx} \quad (1 \le q < 2)$$

i.e.,

$$\int_{-\infty}^{\infty} dx\ \delta(x-x_0) f(x) = f(x_0)$$

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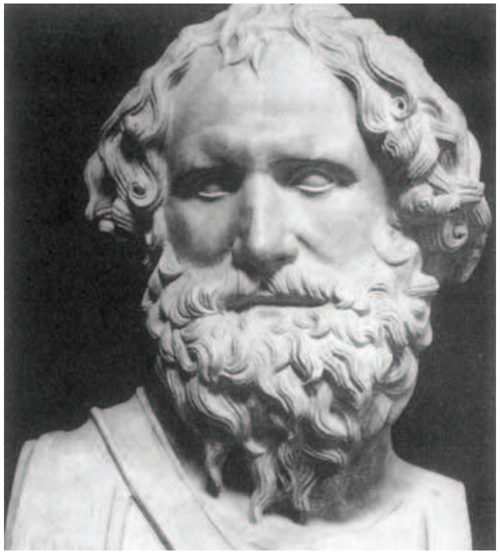
2) New representation of π :



Archimedes

(c. 287 BC – c. 212 BC)

$$\pi = n \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor - 1} (-1)^k \frac{\Gamma\left(n-k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k+2)\Gamma(n-2k)}, \quad \forall n \in \mathbb{N}$$

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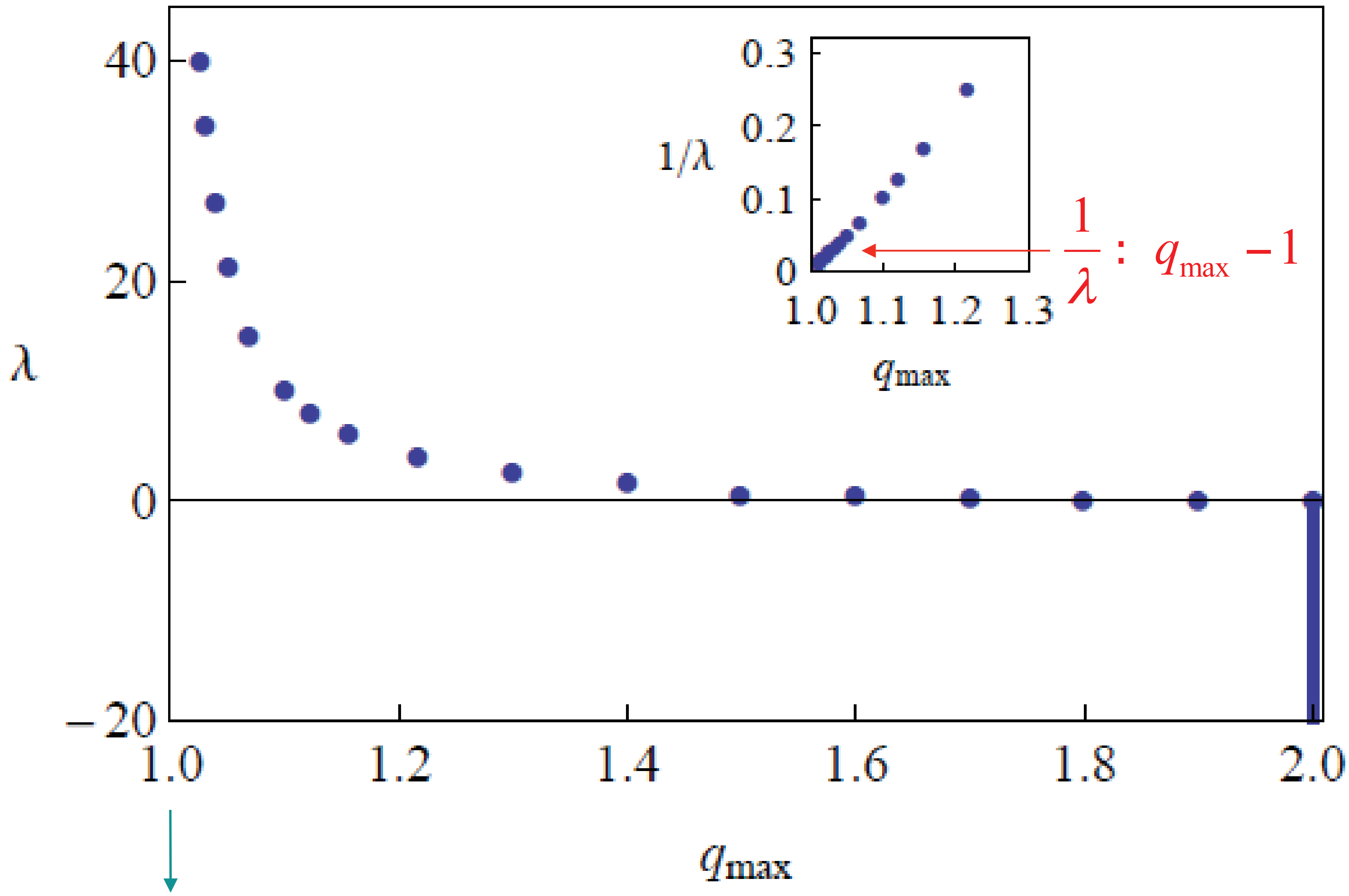
standard
Dirac delta

$$f(x) \sim A|x|^{\lambda} \quad (|x| \to \infty;\ \lambda \in R)$$



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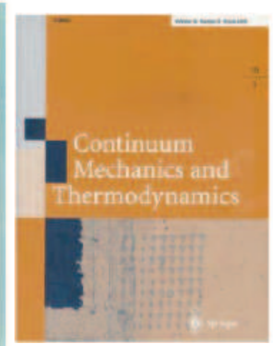
BOOKS AND SPECIAL ISSUES ON NONEXTENSIVE STATISTICAL MECHANICS

1999

2001

2002

2002

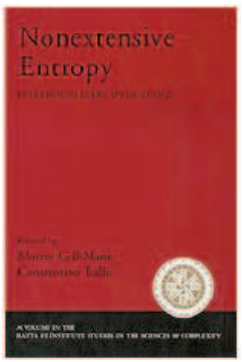


2004

2004

2004

2004



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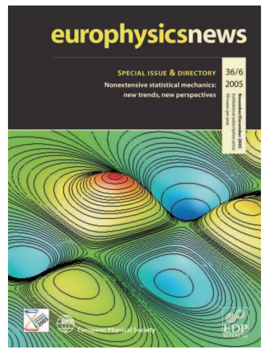
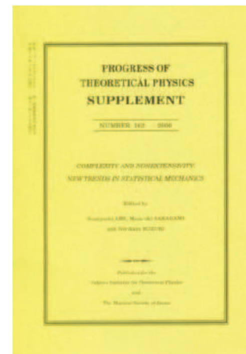


2005

2006



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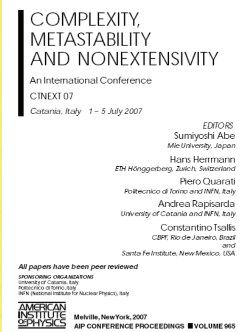

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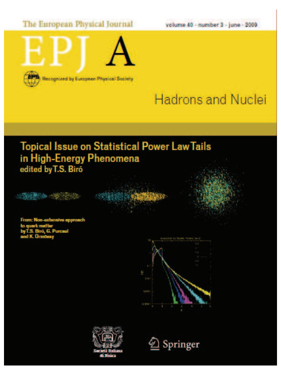
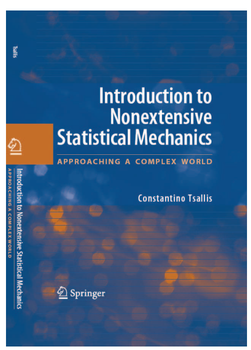


2009

2009



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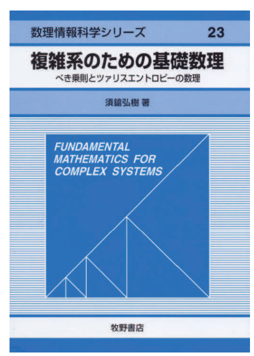

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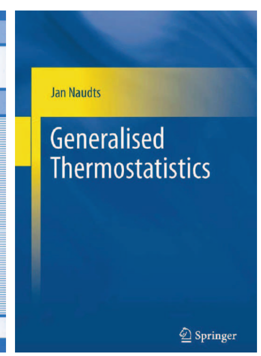
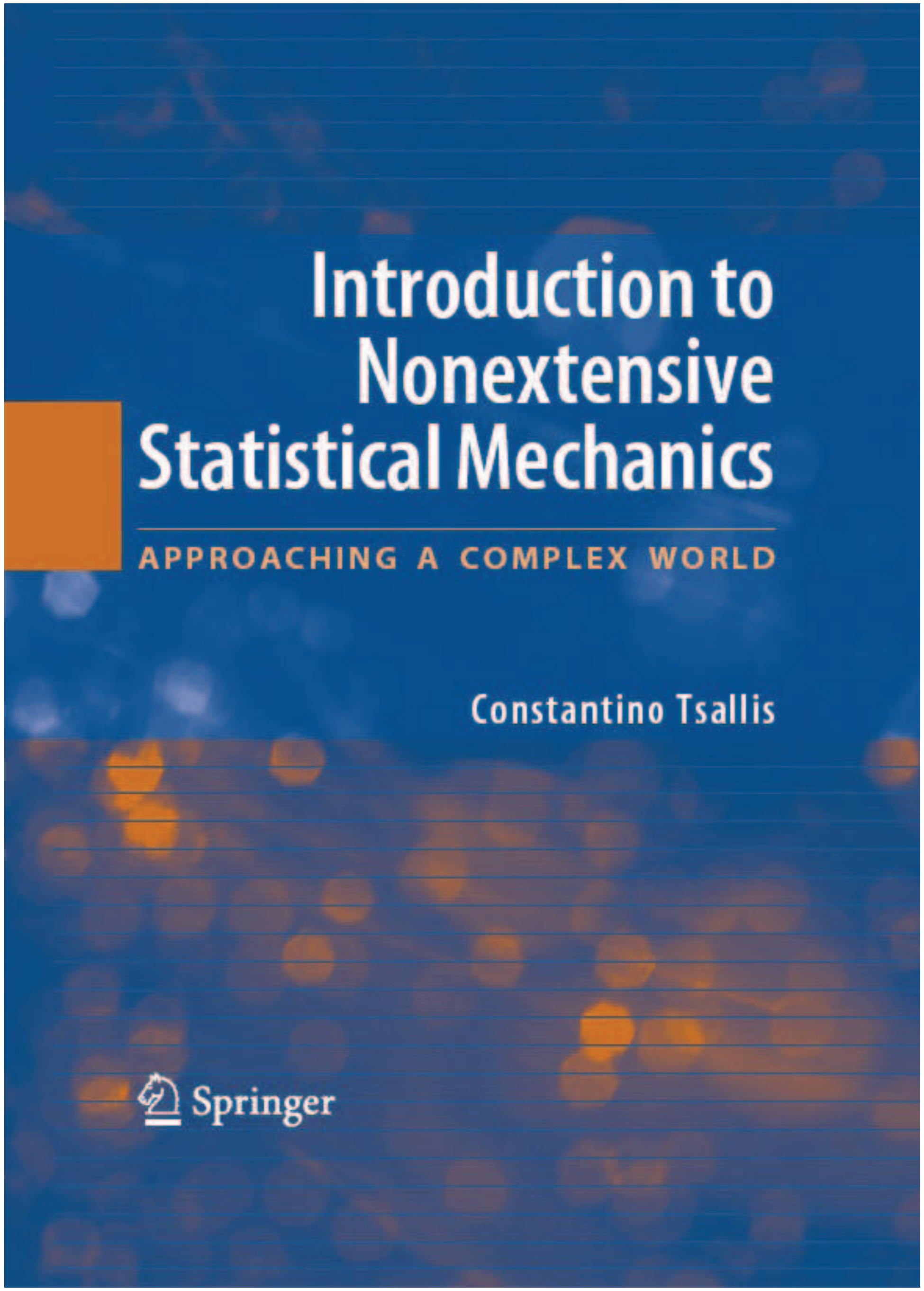


2011



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Constantino Tsallis
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