Convergence results and sharp estimates for the voter model interfaces

Glauco Valle
(IM-UFRJ)

Joint work with: S. Belhaouari, T. Mountford and Rongfeng Sun

The one-dimensional voter model

Interacting Particle Systems with configuration space $\{0,1\}^{\mathbb{Z}}$ and generator acting on local functions $F : \Omega \to \mathbb{R}$ as

$$\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} p(y-x)\mathbf{1}\{\eta(x)\neq\eta(y)\}[F(\eta^x)-F(\eta)]$$

for every $\eta\in\Omega$, where

$$\eta^x(z) = \begin{cases} \eta(z), & \text{if } z\neq x \\ 1-\eta(z), & \text{if } z = x\,. \end{cases}$$

and $p(\cdot)$ is a irreducible symmetric probability kernel on $\mathbb{Z}$ with finite absolute first moment.

(*) There exists $\gamma \geq 1$ such that

$$\sum_{x\in\mathbb{Z}} |x|^\gamma p\,(x) < \infty$$

Let $\eta_{1,0}$ be the Heavyside configuration on $\{0,1\}^{\mathbb{Z}}$, i.e., the configuration:

$$\eta_{1,0}(z) = \begin{cases} 1, & \text{if } z \leq 0 \\ 0, & \text{if } z \geq 1\,, \end{cases}$$

and consider the voter model $(\eta_t)_{t\geq 0}$ starting at $\eta_{1,0}$. For each time $t > 0$, let

$$r_t = \sup\{x : \eta_t(x) = 1\}$$

and

$$l_t = \inf\{x : \eta_t(x) = 0\},$$

which are respectively the positions of the rightmost 1 and the leftmost 0. We call the voter model configuration between the coordinates l_t and r_t the *voter model interface*, and $r_t - l_t + 1$ is the *interface size*.

Question 1: Is the interface size, i.e., the random variables $(r_t - l_t)_{t\geq 0}$, tight?

Question 2: The second question arises from the observation of Cox and Durrett(1995) that, if $(r_t - \ell_t)_{t\geq 0}$ is tight, then the finite-dimensional distributions of

$$\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0} \quad \text{and} \quad \left(\frac{l_{tN^2}}{N}\right)_{t\geq 0}$$

converge to those of a Brownian motion with speed $\sigma := \left(\sum_{z\in\mathbb{Z}} z^2 p(z)\right)^{1/2}$. Do the distributions on $D([0,+\infty),\mathbb{R})$ of

$$\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0} \quad \text{and} \quad \left(\frac{l_{tN^2}}{N}\right)_{t\geq 0}$$

converge weakly to a one-dimensional σ-speed Brownian Motion as $N \to \infty$, i.e, $(\sigma B_t)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian Motion?

Question 3: Let $\{\Theta_x : x \in \mathbb{Z}\}$ be the group of translations on $\{0,1\}^{\mathbb{Z}}$, i.e., $(\eta \circ \Theta_x)(y) = \eta(y + x)$ for every $x \in \mathbb{Z}$ and $\eta \in \Omega$. The third question concerns the equilibrium distribution of $(\eta_t \circ \Theta_{\ell_t})_{t \geq 0}$ when such an equilibrium exists. It was observed by Cox and Durrett (1995) that the question of tightness of $(r_t - \ell_t)_{t \geq 0}$ could be recast as a question concerning the countable state space Markov chain $(\eta_t \circ \Theta_{\ell_t} | \mathbb{N})_{t \geq 0}$ on

$$\left\{ \xi \in \{0,1\}^{\mathbb{N}} : \sum_{x \geq 1} \xi(x) < \infty \right\} .$$

The family $(r_t - \ell_t)_{t \geq 0}$ is tight if and only if $(\eta_t \circ \Theta_{\ell_t} | \mathbb{N})_{t \geq 0}$ is a positive recurrent Markov chain. Cox and Durrett also noted that if $(\eta_t \circ \Theta_{\ell_t})_{t \geq 0}$ was indeed positive recurrent with equilibrium distribution π, then excluding the trivial nearest neighbor case, the equilibrium has

$$E_\pi[\sup\{x : \xi(x) = 1\}] = \infty .$$

How does the tail distribution of the interface size under the equilibrium distribution decays?

Comments on question 1:

$\gamma \geq 2 \Rightarrow$ The interface is tight (Belhaouari and Mountford and previously Cox and Durrett for $\gamma \geq 3$).

$\sum_{x\in\mathbb{Z}} |x|^\gamma p\,(x) = \infty$ for some $\gamma \in (0,2) \Rightarrow$ The interface is not tight (Belhaouari and Mountford).

Theorem 1: For the one-dimensional voter model

(i) If $\gamma > 3$, then the path distributions on $D([0,+\infty),\mathbb{R})$ of

$$\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0} \quad \text{and} \quad \left(\frac{l_{tN^2}}{N}\right)_{t\geq 0}$$

converge weakly to a one-dimensional σ-speed Brownian Motion.

(ii) For $(\frac{r_{tN^2}}{N})_{t\geq 0}$ $\left(\text{resp. } (\frac{l_{tN^2}}{N})_{t\geq 0}\right)$ to converge to a Brownian motion, it is necessary that

$$\sum_{x\in\mathbb{Z}} \frac{|x|^3}{\log^{\beta}(|x|\vee 2)} p(x) < \infty$$

for all $\beta > 1$. In particular, if for some $1 \leq \gamma < \tilde{\gamma} < 3$ we have $\sum_x |x|^{\tilde{\gamma}} p(x) = \infty$, then $\{(\frac{r_{tN^2}}{N})_{t\geq 0}\}$ $\left(\text{resp. } (\frac{l_{tN^2}}{N})_{t\geq 0}\right)$ is not a tight family in $D([0,+\infty),\mathbb{R})$, and hence cannot converge in distribution to a Brownian motion.

Remark: item (i) extends a recent result of Newman, Ravishankar and Sun (2005), in which they obtained the same result for $\gamma \geq 5$ as a corollary of the convergence of systems of coalescing random walks to the so-called *Brownian web* under a finite fifth moment assumption.

Theorem 2: Let $\mathcal{X}_1$ denote the random set of continuous time rate 1 coalescing random walk paths with one walker starting from every point on the space-time lattice $\mathbb{Z}\times\mathbb{R}$, where the random walk increments all have distribution $p(\cdot)$. Let $\mathcal{X}_\delta$ denote $\mathcal{X}_1$ diffusively rescaled, i.e., scale space by δ/σ and time by δ^2. If $\gamma > 3$, then in the topology of the Brownian web, $\mathcal{X}_\delta$ converges weakly to the standard Brownian web $\bar{\mathcal{W}}$ as $\delta \to 0$. A necessary condition for this convergence is again $\sum_{x\in\mathbb{Z}} \frac{|x|^3}{\log^\beta(|x|\vee 2)} p(x) < \infty$ for all $\beta > 1$.

Theorem 3: Take $2 < \gamma < 3$ and fix $0 < \theta < \frac{\gamma-2}{\gamma}$. For $N \geq 1$, let $(\eta_t^N)_{t\geq 0}$ be described as the voter model according to the same Harris system and also starting from $\eta_{1,0}$ except that a flip from 0 to 1 at a site x at time t is suppressed if it results from the "influence" of a site y with $|x-y| \geq N^{1-\theta}$ and $[x \wedge y, x \vee y] \cap [r_{t-}^N - N, r_{t-}^N] \neq \phi$, where r_t^N is the rightmost 1 for the process $\eta_\cdot^N$. Then

(i) $\left(\frac{r^N_{tN^2}}{N}\right)_{t\geq 0}$ converge in distribution to a σ-speed Brownian Motion.

(ii) As $N \to \infty$, the integral

$$\frac{1}{N^2}\int_0^{TN^2} I_{r_s^N \neq r_s} ds$$

tends to 0 in probability for all $T > 0$.

Theorem 4: For the non-nearest neighbor one-dimensional voter model defined as above

(i) If $\gamma \geq 2$, then there exists $C_1 > 0$ such that for all $M \in \mathbb{N}$

$$\pi\{\xi : \Gamma(\xi) \geq M\} \geq \frac{C_1}{M}.$$

(ii) If $\gamma > 3$, then there exists $C_2 > 0$ such that for all $M \in \mathbb{N}$

$$\pi\{\xi : \Gamma(\xi) \geq M\} \leq \frac{C_2}{M}.$$

(iii) Let $\alpha = \sup\{\gamma : \sum_{x \in \mathbb{Z}} |x|^\gamma p(x) < \infty\}$. If $\alpha \in (2, 3)$, then

$$\limsup_{n \to \infty} \frac{\log \pi\{\xi : \Gamma(\xi) \geq n\}}{\log n} \geq 2 - \alpha.$$

Furthermore, there exist choices of $p(\cdot) = p_\alpha(\cdot)$ with $\alpha \in (2, 3)$ and

$$\pi\{\xi : \Gamma(\xi) \geq n\} \geq \frac{C}{n^{\alpha-2}}$$

for some constant $C > 0$.

(i) For every $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < ... < t_n$ in $[0, \infty)$ the finite-dimensional distribution

$$\left(\frac{r_{t_2 N^2} - r_{t_1 N^2}}{\sigma N \sqrt{t_2 - t_1}}, \ldots, \frac{r_{t_n N^2} - r_{t_{n-1} N^2}}{\sigma N \sqrt{t_n - t_{n-1}}}\right)$$

converges weakly to a centered n-dimensional Gaussian vector of covariance matrix equal to the identity. Moreover the same holds if we replace r_t by l_t. (True for $\gamma \geq 2$ by Cox and Durrett and Bransom and Mountford)

(ii) For every $\epsilon > 0$ and $T > 0$

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathrm{P}\left[\sup_{\substack{|t-s|<\delta \\ s,t \in [0,T]}} \left|\frac{r_{tN^2} - r_{sN^2}}{N}\right| > \epsilon\right] = 0\,.$$

In particular if the finite-dimensional distributions of $\left(\frac{r_{tN^2}}{N}\right)_{t \geq 0}$ are tight, we have that the path distribution is also tight and every limit point is concentrated on continuous paths. The same holds if we replace r_t by l_t.

By the Markov property, recurrence, the right continuity of paths and the fact that the voter model is attractive, we have that (ii) is a consequence of: for all $\epsilon > 0$

$$\limsup_{\delta \to 0} \delta^{-1} \limsup_{N \to +\infty} \mathsf{P}\left[\sup_{0 \le t \le N^2\delta} |r_t| \ge \epsilon N\right] = 0\,.$$

It is sufficient to show that

$$\limsup_{\delta \to 0} \delta^{-1} \limsup_{N \to +\infty} \mathsf{P}\left[\sup_{0 \le t \le N^2\delta} r_t \ge \epsilon N\right] = 0\,. (*)$$

Indeed, to see that

$$\limsup_{\delta \to 0} \delta^{-1} \limsup_{N \to +\infty} \mathsf{P}\left[\inf_{0 \le t \le N^2\delta} r_t \le -\epsilon N\right] = 0\,,$$

note that $r_t \ge l_t - 1$, and

$$\limsup_{\delta \to 0} \delta^{-1} \limsup_{N \to +\infty} \mathsf{P}\left[\inf_{0 \le t \le N^2\delta} l_t \le -\epsilon N\right] = 0\,,$$

is equivalent to (*) by interchanging the 0's and 1's in the voter model.

Equation (*) is equivalent to

$$\lim_{\delta\to 0} \delta^{-1} \limsup_{N\to+\infty} \mathsf{P}\Big[\zeta_t^t([\epsilon N,+\infty))\cap(-\infty,0]\neq\phi \text{ for some } t\in[0,\delta N^2]\Big] = 0\,.$$

Taking $R := R(\delta, N) = \sqrt{\delta}N$ and $M = \epsilon/\sqrt{\delta}$:

$$\lim_{M\to+\infty} M^2 \limsup_{R\to+\infty} \mathsf{P}\Big[\zeta_t^t([MR,+\infty))\cap(-\infty,0]\neq\phi \text{ for some } t\in[0,R^2]\Big] = 0,$$

which means that we have to estimate the probability that no dual coalescing random walk starting at a site in $[MR, +\infty)$ at a time in the interval $[0, R^2]$ arrive at time $t = 0$ at a site to the left of the origin.

Proposition: If $\gamma > 3$, then for $R > 0$ sufficiently large and $2^b \leq M < 2^{b+1}$, for some $b \in \mathbb{N}$ the probability

$$\mathsf{P}\left[\zeta_t^t([MR,+\infty)) \cap (-\infty, 0] \neq \phi : \text{ for some } t \in [0, R^2]\right]$$

is bounded above by a constant times

$$\sum_{k\geq b}\left\{\frac{1}{2^{2k}R^{\frac{\gamma-3}{2}}} + e^{-c2^k} + 2^k R^4 e^{-c2^{k(1-\beta)}R^{\frac{(1-\beta)}{2}}} + 2^k e^{-c2^{2k}}\right\}$$

for some $c > 0$ and $0 < \beta < 1$.

For the rescaled interface boundary $\frac{r_{tN^2}}{N}$ to converge to a σ-speed Brownian motion, we must have

$$\lim_{t\to 0}\limsup_{N\to\infty} \mathrm{P}\left[\sup_{0\le s\le t}\frac{r_{tsN^2}}{N} > \epsilon\right] = 0.$$

This is equivalent to

$$\lim_{t\to 0}\limsup_{N\to\infty} \mathrm{P}\left\{\zeta_s^s([\epsilon N, +\infty)) \cap (-\infty, 0] \neq \phi \quad \text{for some } s \in [0, tN^2]\right\} = 0.$$

Since random walk jumps originating from

$$(-\infty, -\epsilon N] \cup [\epsilon N, +\infty)$$

which crosses level 0 in one step occur as a Poisson process with rate $\sum_{k=\epsilon N}^{\infty} F(k)$ where $F(k) = \sum_{|x|\ge k} p(x)$, previous condition implies that

$$\limsup_{N\to\infty} N^2 \sum_{k=\epsilon N}^{\infty} F(k) \le C < +\infty.$$

In particular,

$$\sup_{N\in\mathbb{Z}^+} N^2 \sum_{k=N}^{\infty} F(k) \le C_\epsilon < +\infty.$$

Let $H(y) = y^3 \log^{-\beta}(y \vee 2)$ for some $\beta > 1$.

Let $H^{(1)}(k) = H(k) - H(k-1)$

and $H^{(2)}(k) = H^{(1)}(k) - H^{(1)}(k-1)$.

$k_0 \in \mathbb{Z}^+$, such that $0 < H^{(2)}(k) < 8k \log^{-\beta} k$ for $k > k_0$.

$G(k) = \sum_{i=k}^{\infty} F(i)$ and thus $G(k) \le \frac{C_\epsilon}{k^2}$ for all $k \in \mathbb{Z}^+$.

Then

$$\sum_{k\in\mathbb{Z}} H(|k|)p(k) = \sum_{k=1}^{\infty} 2H(k)p(k)$$

is equal to

$$\sum_{k=1}^{k_0-1} 2H(k)p(k) + H(k_0)F(k_0)$$
$$+H^{(1)}(k_0+1)G(k_0+1) + \sum_{k=k_0+2}^{\infty} H^{(2)}(k)G(k)$$

which is bounded by

$$\sum_{k=1}^{k_0-1} 2H(k)p(k) + H(k_0)F(k_0)$$
$$+H^{(1)}(k_0+1)G(k_0+1) + \sum_{k=k_0+2}^{\infty} \frac{8k}{\log^{\beta} k} \cdot \frac{C_\epsilon}{k^2}$$
$$< \infty$$

We assume that $2 < \gamma < 3$ and we fix $0 < \theta < \frac{\gamma-2}{\gamma}$.

Lemma: For almost surely all realizations of the Harris system in the time interval $[0, \delta N^2]$ with $\sup_{0 \leq t \leq \delta N^2} r_t^N \geq \epsilon N$ for some $0 < \epsilon < 1$, there exists a dual backward random walk starting from some site in $\{\mathbb{Z} \cap [\epsilon N, +\infty)\} \times [0, \delta N^2]$ which attains the left of the origin before time 0 in the voter model by making no jumps of size greater than or equal to $N^{1-\theta}$.

Take $0 < \sigma < \theta$ and let $\epsilon' := \frac{(1-\theta)(3-\gamma)}{\sigma}$.

$$\sum_{|x| \leq N^{1-\theta}} |x|^{3+\epsilon'} p(x) \leq C N^{(1-\theta+\sigma)\epsilon'}.$$

The estimate required here is the same as in the proof of Theorem 1, except that as we increase the index N, the random walk kernel also changes and its $(3+\epsilon')$th-moment increases as $CN^{(1-\theta+\sigma)\epsilon'}$. Denote by ζ^N the system of coalescing random walks with jumps larger than or equal to $N^{1-\theta}$ suppressed, and recall that $R = \sqrt{\delta}N$ and $M = \epsilon/\sqrt{\delta}$ in our argument.

Proposition: For $R > 0$ sufficiently large and $2^b \leq M < 2^{b+1}$ for some $b \in \mathbb{N}$, the probability

$$\mathsf{P}\left\{\zeta_t^{N,t}([MR,+\infty)) \cap (-\infty,0] \neq \phi : \text{ for some } t \in [0,R^2]\right\}$$

is bounded above by a constant times

$$\sum_{k \geq b}\left\{\frac{1}{2^{2k}\delta\epsilon' R^{\frac{(\theta-\sigma)\epsilon'}{2}}} + e^{-c2^k} + 2^k R^4 e^{-c2^{k(1-\beta)}R^{\frac{(1-\beta)}{2}}} + 2^k e^{-c2^{2k}}\right\}$$

for some $c > 0$ and $0 < \beta < 1$.

Since $(\eta_t \circ \Theta_{\ell_t} | \mathbb{N})_{t \geq 0}$ is a positive recurrent Markov chain on $\tilde{\Omega}$, by usual convergence results, we have to show that starting from the Heavy-side configuration for every t and M sufficiently large

$$P(r_t - l_t \geq M) \geq \frac{C}{M},$$

for some $C > 0$ independent of M and t.

Fix $\lambda > 0$,

$$\begin{aligned} &P(r_t - l_t \geq M,\ r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M) \\ &\quad = P(r_t - l_t \geq M | r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M) \times \\ &\qquad\qquad P(r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M), \end{aligned}$$

which by tightness is bounded below by

$$\frac{1}{2} P(r_t - l_t \geq M | r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M)$$

for M sufficiently large.

Let $(X_t^{-M})_{t\geq 0}$ and $(X_t^{M})_{t\geq 0}$ be two independent random walks starting respectively at $-M$ and M at time 0 with transition probability $p(\cdot)$. Denote $Z_t^M = X_t^M - X_t^{-M}$. For every set $A \subset \mathbb{Z}$, let τ_A be the stopping time

$$\inf\{t \geq 0 : Z_t^M \in A\}\,.$$

If $A = \{x\}$, we denote τ_A simply by τ_x. Then by duality and the Markov property after translating the system to have the leftmost 0 at the origin by time $t - \lambda M^2$ we obtain that

$$P(r_t - l_t \geq 2M | r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq 2M)$$
$$\geq P(\tau_0 > \lambda M^2; X_{\lambda M^2}^{-M} \geq M; X_{\lambda M^2}^{M} \leq -M)\,.$$

Lemma: If $p(\cdot)$ is a non-nearest neighbor transition probability and has zero mean and finite second moment, then we can take λ sufficiently large such that for some $C > 0$ independent of M and for all M sufficiently large,

$$P(\tau_0 > \lambda M^2; X_{\lambda M^2}^{-M} \geq M; X_{\lambda M^2}^{M} \leq -M) \geq \frac{C}{M}.$$

Let $A_s(M,k,x)$ be the event

$$\{\tau_0^{x,x+k} > \lambda M^2 - s;\ X^{x+k}_{\lambda M^2-s} \geq M;\ X^{x}_{\lambda M^2-s} \leq -M\},$$

where as before, for every x and y, $(X_t^x)_{t\geq 0}$ and $(X_t^y)_{t\geq 0}$ denote two independent random walks starting respectively at x and y with transition probability $p(\cdot)$, and

$$\tau_0^{x,x+k} = \inf\{t \geq 0 : X_t^{x+k} - X_t^x = 0\}.$$

Lemma: Let $K \in \mathbb{N}$ be fixed. For all $l \in \mathbb{N}$ sufficiently large, there exists some $C > 0$ such that for all $s \leq \lambda M^2/2$, $|x| < lM$ and $0 < k \leq K$, and M sufficiently large

$$P(A_s(M,k,x)) > \frac{C}{M}.$$

It suffices to show that, for every $M > 0$, if t is sufficiently large, then

$$\mathsf{P}(r_t - l_t \geq M) \leq \frac{C}{M}$$

for some $C > 0$ independent on M and t.

Fix $N \in \mathbb{N}$ and assume $M = 2^N$. In the following t will be $>> 2^{2N}$. Let $\Delta_t(s)$, for $s < t$, be the event that a crossing of two dual coalescing random walks starting at time t (in the voter model) occurs in the dual time interval $(s, t]$ and by the dual time t they are on opposite sides of the origin,.

From the estimates in Cox and Durrett, one can show that $\mathsf{P}(\Delta_t(s)) \leq C/\sqrt{s}$, if we have that $\mathsf{P}(0 \in \zeta_s^s(\mathbb{Z})) \leq C/\sqrt{s}$, which holds if $p(\cdot)$ has finite second moment. Therefore, all we have to show is that

$$\mathsf{P}\left(\{r_t - l_t \geq 2^N\} \cap (\Delta_t(4^N))^c\right) \leq \frac{C}{2^N}$$

for some C independent of t and N.

The event

$$\{r_t - l_t \geq 2^N\} \cap (\Delta_t(4^N))^c$$

is a subset of $\cup_{k=0}^N V_k^N$ where V_k^N is the event that there exists $x, y \in \mathbb{Z}$ with $y - x \geq 2^N$ such that, for the coalescing walks $X_s^{x,t}$ and $X_s^{y,t}$,

(i) $X_s^{x,t} < X_s^{y,t}$ for every $0 \leq s \leq 4^{k-1}$;

(ii) There exists $s \in (4^{k-1}, 4^k]$ with $X_s^{x,t} > X_s^{y,t}$;

(iii) $X_t^{x,t} > 0$ and $X_t^{y,t} \leq 0$.

For $k = 0$ we replace 4^{k-1} by 0. We obtain suitable bounds on V_k^N which will enable us to conclude that $\sum_{k=0}^N P(V_k^N) \leq \frac{C}{2^N}$.

Fix $0 \le k \le N$. For $0 \le s \le t$ and $y \in \mathbb{Z}$, we call

$$R_y(s) = \begin{cases} \sup_{x\in\mathbb{Z}}\{|x-y| : X_s^{x,t} = y\} & , \exists x : X_s^{x,t} = y \\ 0 & , \text{ otherwise} \end{cases}$$

the *range* of the coalescing random walk at $(s, y) \in (0, t] \times \mathbb{Z}$. Obviously V_k^N is contained in the event that there exists x, y in $\zeta^t_{4^{k-1}}(\mathbb{Z})$ with $x < y$ such that

(i) $R_x(4^{k-1}) + R_y(4^{k-1}) + |y - x| \ge 2^N$;

(ii) There exists $s \in (4^{k-1}, 4^k]$ with $X^{x,t-4^{k-1}}_{s-4^{k-1}} > X^{y,t-4^{k-1}}_{s-4^{k-1}}$;

(iii) $X^{x,t-4^{k-1}}_{t-4^{k-1}} > 0$, $X^{y,t-4^{k-1}}_{t-4^{k-1}} \le 0$,

which we denote by $\tilde{V}_k^N$.

We call the crossing between two coalescing random walks a *relevant crossing* if it satisfies conditions (i) and (ii) in the definition of $\tilde{V}_k^N$ up to the time of the crossing. We are interested in the density of relevant crossings between random walks in the time interval $(4^{k-1}, 4^k]$ and (as is also relevant) the size of the overshoot. We consider separately three cases:

(i) The random walks at time 4^{k-1} are at $x < y$ with $|x-y| \le 2^{k-1}$ (so it is "reasonable" to expect the random walks to cross in the time interval $(4^{k-1}, 4^k]$, and either $R_x(4^{k-1})$ or $R_y(4^{k-1})$ must exceed 2^{N-2}).

(ii) The random walks are separated at time 4^{k-1} by at least 2^{k-1} but no more than 2^{N-1} (so either $R_x(4^{k-1})$ or $R_y(4^{k-1})$ must exceed 2^{N-2}).

(iii) The random walks are separated at time 4^{k-1} by at least 2^{N-1}. In this case we disregard the size of the range.

We estimate the relevant crossing densities and overshoot size in cases (i), (ii) and (iii) above. We first estimate the expectation of the overshoot between two random walks starting at $x < y$ at time 4^{k-1} restricted to the event that: $x, y \in \zeta^t_{4^{k-1}}(\mathbb{Z})$, R_x and R_y are compatible with $y - x$ as stated in cases (i)–(iii), and the two walks cross before time 4^k. Then we fix a site $x \in \mathbb{Z}$ and summing over $y \in \mathbb{Z}$, we obtain that the total expected overshoot associated with relevant crossings in the time interval $(4^{k-1}, 4^k]$ involving $(x, 4^{k-1})$ and $(y, 4^{k-1})$ for all possible $y \in \mathbb{Z}$ is bounded by

$$C\left(\frac{1}{2^{N(1+\epsilon)}} + e^{-c2^{N(1-\beta)}} + \frac{e^{-c4^{N-k}}}{2^k}\right),$$

where $\beta \in (0,1)$ is fixed.

Lemma: For every $0 < \beta < 1$, there exists $c, C \in (0, \infty)$ so that for every $k \in \mathbb{N}$ and $m \geq 1$, the density of $y \in \zeta^t_{4^k}(\mathbb{Z})$ such that on the (dual) time interval $(4^k, 4^{k+1}]$ the corresponding random walk distances itself from y by $m2^k$ is bounded by

$$\frac{C}{2^k}\left(e^{-c(m2^k)^{1-\beta}} + e^{-cm^2} + \frac{1}{m^{3+\epsilon}2^{k(1+\epsilon)}}\right).$$

As a corollary, we have

Lemma: For every $0 < \beta < 1$, there exists $c, C \in (0, \infty)$ so that for every $k \in \mathbb{N}$ and $m \geq 1$, the density of $y \in \zeta^t_{2^{2k}}(\mathbb{Z})$ whose range is greater than $m2^k$ is bounded by

$$\frac{C}{2^k}\left(2^k e^{-c(m2^k)^{1-\beta}} + e^{-cm^2} + \frac{1}{m^{3+\epsilon}2^{k(1+\epsilon)}}\right).$$

We say a d-crossover ($d \in \mathbb{N}$) occurs at site $x \in \mathbb{Z}$ at time $s \in (4^{k-1}, 4^k]$ if at this time (dual time, for coalescing random walks) a relevant crossing occurs leaving particles at sites x and $x+d$ immediately after the crossing. We denote the indicator function for such a crossover by $I_k(s,x,d)$. By translation invariance, the distribution of $\{I_k(s,x,d)\}_{s\in(4^{k-1},4^k]}$ is independent of $x \in \mathbb{Z}$.

Let X_s^x and X_s^{x+d} be two independent random walks starting at x and $x+d$ at time 0, and let $\tau_{x,x+d} = \inf\{s : X_s^x = X_s^{x+d}\}$. Then $P(\tilde{V}_k^N)$ is bounded above by

$$\sum_{d\in\mathbb{N}} \sum_{x\in\mathbb{Z}} E\left[\int_{4^{k-1}}^{4^k} ds I_k(s,x,d) \times \right.$$
$$\left. P\left(X_{t-s}^x \le 0 < X_{t-s}^{x+d}, \tau_{x,x+d} > t-s\right)\right]$$
$$= \sum_{d\in\mathbb{N}} \left\{ E\left[\int_{4^{k-1}}^{4^k} I_k(s,0,d) ds\right] \times \right.$$
$$\left. \sum_{x\in\mathbb{Z}} P\left(X_{t-s}^x \le 0 < X_{t-s}^{x+d}, \tau_{x,x+d} > t-s\right)\right\}.$$

If we know that

$$\sum_{x\in\mathbb{Z}} P\left(X^x_{t-s} \leq 0 < X^{x+d}_{t-s}, \tau_{x,x+d} > t-s\right) \leq Cd$$

for some $C > 0$ independent of k, d, s, t and N, and

$$E\left[\sum_{d\in\mathbb{N}} d \int_{4^{k-1}}^{4^k} I_k(s,0,d)ds\right]$$

is dominated by

$$C\left(\frac{1}{2^{N(1+\epsilon)}} + e^{-c2^{N(1-\beta)}} + \frac{e^{-c4^{N-k}}}{2^k}\right).$$

Therefore

$$\sum_{k=0}^{N} P(\tilde{V}_k^N)$$

is bounded above by

$$\sum_{k=0}^{N}\left(\frac{1}{2^{N(1+\epsilon)}} + e^{-c2^{N(1-\beta)}} + \frac{e^{-c4^{N-k}}}{2^k}\right) \leq \frac{C'}{2^N}$$

for some $C' > 0$ uniformly over all large t and N and we are done.

If we denote $Z^d_{s'} = X^{x+d}_{s'} - X^x_{s'}$, $(Z^d_{s'})^+ = Z^d_{s'} \vee 0$ and $\tau_0 = \inf\{s' : Z^d_{s'} = 0\}$, then by translation invariance, it is not difficult to see that

$$\sum_{x \in \mathbb{Z}} P\left(X^x_{t-s} \le 0 < X^{x+d}_{t-s}, \tau_{x,x+d} > t-s\right)$$

is equal to

$$E[(Z^d_{t-s})^+, \tau_0 > t-s] \le Cd,$$

where the inequality with $C > 0$ uniform over d and t is a standard result for random walks.

If $\gamma \geq 2$, then the voter model interface evolves as a positive recurrent chain (Bransom and Mountford), and hence the equilibrium distribution π exists and $\pi\{\xi_0\} > 0$ where ξ_0 is the trivial interface of the Heavyside configuration $\eta_{1,0}$. Let ξ_t denote the interface configuration at time t starting with ξ_0, and let ν denote its distribution. Then

$$\pi\{\xi : \Gamma(\xi) \geq n\} > \pi\{\xi_0\}\nu\{\Gamma(\xi_t) \geq n\}$$

for all $t > 0$. It suffices to show

$$\limsup_{n\to\infty} \frac{\log \nu\{\Gamma(\xi_{n^2}) \geq n\}}{\log n} \geq 2 - \alpha.$$

Lemma: Let X_t be a centered continuous time one-dimensional random walk starting at the origin and with finite $3+\epsilon$ moment for some $\epsilon > 0$. Then for every $0 < \beta < 1$, there exists $c, C > 0$ such that

$$\mathbb{P}\left(\sup_{t \leq T} |X_t| \geq M\right) \leq C\left(e^{-cM^{1-\beta}} + e^{-\frac{cM^2}{T}} + \frac{T}{M^{3+\epsilon}}\right)$$

for all $T, M > 0$.

Lemma 1: Let X_t^x and X_t^y be two independent identically distributed continuous time homogeneous random walks with finite second moments starting from positions x and y at time 0. Let $\tau_{x,y} = \inf\{t > 0 : X_t^x = X_t^y\}$ be the first meeting time of the two walks. Then there exists $C_0 > 0$ such that

$$P(\tau_{x,y} > T) \leq \frac{C_0}{\sqrt{T}}|x-y|$$

for all x, y and $T > 0$.

Lemma: Given a system of $2J$ coalescing random walks indexed by their starting positions $\{x_1^{(1)}, x_2^{(1)},\ \ldots,\ x_1^{(J)}, x_2^{(J)}\}$ at time 0, if

$$x_1^{(1)} < x_2^{(1)} < \cdots < x_1^{(i)} < x_2^{(i)} < \cdots < x_1^{(J)} < x_2^{(J)},$$

and $\sup_i |x_1^{(i)} - x_2^{(i)}| \leq M$ for some $M > 0$, then for any fixed time $T > C_0^2 M^2$ with C_0 satisfying Lemma 1, the number of coalesced walks by time T stochastically dominates the sum of J independent Bernoulli random variables $\{Y_1, \ldots, Y_J\}$, each with parameter $1 - C_0 M/\sqrt{T}$. In particular the probability that the number of coalesced particles by time T is smaller than N is bounded above by

$$P\left(\sum_{i=1}^{J} Y_i \leq N\right).$$

Proposition: Let $\frac{1}{2} < p < 1$ be fixed. Consider a system of coalescing random walks starting with at most γL particles inside an interval of length L at time 0. Let $K_0 = \frac{64C_0^2}{(2p-1)^4}$, where C_0 is as in Lemma 1. If $\gamma L \geq \frac{8}{2p-1}$, then there exist constants C, c depending only on p such that, the probability that the number of particles alive at time $T = \frac{K_0}{\gamma^2}$ is greater than $p\gamma L$ is bounded above by $Ce^{-c\gamma L}$.

Lemma: In the system of backward coalescing random walks $\{X^{x,s}\}_{(x,s)\in\mathbb{Z}\times\mathbb{R}}$ dual to the voter model, assume the random walk increment distribution $p(\cdot)$ has finite $3+\epsilon$ moment. Then there exist $C>0$ depending only on $p(\cdot)$, such that for all $K \geq 1$,

$$P\Big\{ \text{ for some } (x,s) \in [2^k R, 2^{k+1} R] \times [0, R^2],$$
$$|X_u^{x,s} - x| \geq \frac{2^k R}{(\log R)^2}$$
$$\text{for some } 0 \leq u \leq s - K\lfloor \frac{s-1}{K} \rfloor \Big\}$$

is bounded above by

$$\frac{CK(\log R)^{2(3+\epsilon)}}{2^{2k+3\epsilon} R^{\epsilon}}$$

for all R sufficiently large.

Lemma: Let $\zeta_t^{\mathbb{Z}}$ be the process of coalescing random walks starting from $\mathbb{Z}$ at time 0 where all random walk increments are distributed according to a transition probability $p(\cdot)$ with finite second moment. Then for all $t > 0$

$$\mathbb{P}(0 \in \xi_t^{\mathbb{Z}}) \leq \frac{C}{\sqrt{t}}$$

for some $C > 0$.