# Multiscale stochastic control with state constraints

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# Multiscale challenges in hydrothermal systems

## RENEWABLE PENETRATION

Increasing penetration of intermittent wind and solar generation fundamentally changes the operation and planning of electric grids, traditionally hydrothermal.

## TWO-SCALE UNCERTAINTY

The system must handle high-frequency intraday noise in wind and solar generation while hedging against long-term seasonal hydrological variability.

## MAIN CHALLENGE

Interacting fast and slow dynamics lead to a nonlinear multiscale stochastic optimization problem.
The aim is to reduce opetation costs.

## Can a constrained slow-fast system be replaced by a rigorous lower-dimensional control problem?

- The slow state is controlled and must remain inside a fixed domain.
- A hidden fast variable changes the observed drift, diffusion and cost.
- The reduced model must preserve control, reflection and boundary costs.

# The two companion papers

OVERVIEW

1 **A. Calixto, B. Costa, G. Valle**

Stochastic optimal control with control-dependent diffusion and state constraints: a degenerate elliptic approach.

*Math. Control and Rel. Fields., online (early access), 2026.*

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2 **A. Calixto, B. Costa, G. Valle**

Multiscale Stochastic Control With Domain Restriction in the Slow Variable.

*Journal of Optimization Theory and Applications 209, article 54, 2026.*

# How the two papers fit together



**PAPER 1**

## One reflected controlled diffusion

Existence, continuity, DPP
HJB characterization
Comparison and uniqueness

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**PAPER 2**

## A reflected slow-fast control system

Fast ergodic dynamics
Effective reflected control
Uniform convergence of values

# State, control and coefficient spaces

- $t \in \mathbb{R}_+ = [0, \infty)$ time
- $X \subset \mathbb{R}^d$ is the fixed compact state space; $U \subset \mathbb{R}^m$ is convex and compact (space of allowed control values).
- The drift and diffusion functions are given by

$$\mu_X : X \times U \longrightarrow \mathbb{R}^d, \qquad \sigma_X : X \times U \longrightarrow \mathbb{R}^{d \times d_W}.$$

- An admissible control $(u(t))_{t \geq 0}$ is progressively measurable process with values in $U$; the class of all such controls is denoted by $\mathcal{U}$.
- $X_x^u(t)$ denotes the state at time $t$, starting from $x$ and driven by $u$.
- $(W(t))_{t \geq 0}$ is a $d_W$-dimensional brownian motion.

# Geometry and regularity assumptions

## Domain Geometry

$$\phi \in C_b^2(\mathbb{R}^d), \qquad X = \{\phi \leq 0\}, \quad \text{Int}(X) = \{\phi < 0\}, \quad \partial X = \{\phi = 0\},$$

$$\|D\phi(x)\| = 1 \quad \text{for every } x \in \partial X.$$

## Coefficients

- $\mu_X$ and $\sigma_X$ are continuous in $(x, u)$.
- They are Lipschitz in the state variable, uniformly in $u$: for some $C_0 > 0$,

$$\|\mu_X(x,u) - \mu_X(y,u)\| + \|\sigma_X(x,u) - \sigma_X(y,u)\| \leq C_0\|x - y\|.$$

Consequently, both fields are bounded on the compact set $X \times U$.

# Reflected controlled diffusion

$$\mathrm{d}X_x^u(t) = \mu_X(X_x^u(t), u(t))\,\mathrm{d}t + \sigma_X(X_x^u(t), u(t))\,\mathrm{d}W(t) - D\phi(X_x^u(t))\,\mathrm{d}\ell_x^u(t).$$

The regulator $\ell_x^u$ is singular continuous, adapted and nondecreasing, with $\ell_x^u(0) = 0$, and

$$\ell_x^u(t) = \int_0^t \mathbf{1}_{\partial X}(X_x^u(s))\; \mathrm{d}\ell_x^u(s) \qquad \text{a.s.}$$

- $X_x^u(t) \in X$ for every $t \geq 0$; the state process is continuous and adapted.
- Thus $\mathrm{d}\ell_x^u$ acts only on the process is at the boundary. $-D\phi(X_x^u(t))\,\mathrm{d}\ell_x^u(t)$ points inward and represents a force field that reflects the process back into the domain.
- Under the preceding assumptions, the reflected SDE has a unique strong solution.

# The cost functional

$$J_x^\beta(u) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} L(X_t^u, u_t)\,\mathrm{d}t + \int_0^\infty e^{-\beta t} h(X_t^u)\,\mathrm{d}\ell_x^u(t)\right],$$

$$v^\beta(x) = \inf_{u\in\mathcal{U}} J_x^\beta(u).$$

- $L$ is the running cost accumulated continuously along the controlled trajectory.
- $h$ is the preventive cost at the boundary charged only through increments of the boundary regulator $\ell$.
- $v^\beta$ is called the *value function*.
- The controller minimizes the sum of operating and boundary costs.

- Lions and Sznitman ('84) developed the theory of reflected SDEs, where a boundary regulator keeps the process inside a prescribed domain. Further references include Pardoux and Răşcanu ('14) and Pilipenko ('14).

- Lions and Soner ('84) studied fully nonlinear elliptic equations with state constraints and viscosity boundary conditions. Crandall, Ishii and Lions ('92) established the general viscosity framework for degenerate equations.

- Krylov ('80), Kushner and Dupuis ('92), Fleming and Soner ('06), and Nisio ('15) develop dynamic programming and HJB methods for controlled diffusions.

Our fisrt paper brings these ingredients together when the control acts on a possibly degenerate diffusion. Reflection yields a nonhomogeneous Neumann boundary condition.

# Dynamic programming principle

$$v^\beta(x) = \inf_{u \in \mathcal{U}} \mathbb{E}\Bigg[ \int_0^\tau e^{-\beta t} L(X_t^u, u_t)\, \mathrm{d}t + \int_0^\tau e^{-\beta t} h(X_t^u)\, \mathrm{d}\ell_t^u + e^{-\beta \tau} v^\beta(X_\tau^u) \Bigg].$$

- Pay the running and boundary costs up to a stopping time $\tau$.
- Restart the optimization from the reflected state $X_\tau^u$.
- This identity is the bridge from the stochastic problem to a PDE characterization.

# The HJB–Neumann problem

$$\begin{cases} \beta v^\beta(x) - \inf\limits_{u\in U}\Big\{\langle \mu_X(x,u), Dv^\beta(x)\rangle \\ \qquad\qquad + \frac{1}{2}\operatorname{Tr}(D^2 v^\beta(x)[\sigma_X \sigma_X^\top](x,u)) + L(x,u)\Big\} = 0, \quad x \in X \\ \langle Dv^\beta(x), D\phi(x)\rangle = h(x), \quad x \in \partial X. \end{cases}$$

- The Hamiltonian is fully nonlinear because the control changes the diffusion.
- The diffusion may be degenerate in some directions.
- The boundary cost becomes a nonhomogeneous Neumann condition.

# Why viscosity solutions?

$$v^\beta \in C(X)$$

$$\text{often not } C^2(X)$$

- Test the PDE with smooth functions that touch the value function locally.
- Treat the interior equation and the Neumann condition together.
- The notion is stable under uniform limits – essential for the multiscale result.

# Viscosity solution: the working definition

Consider the problem

$$\begin{cases} F_\varphi(x) := F^\beta\Big(x, \varphi(x), D\varphi(x), D^2\varphi(x)\Big) = 0, & x \in X \\ \Gamma_\varphi(x) := \Gamma(x, D\varphi(x)) = 0, & x \in \partial X. \end{cases}$$

**SUBSOLUTION**

$w \in USC(X)$; every $\varphi \in C^3(X)$ touching $w$ from above at $x_0$ must satisfy

$$\begin{cases} F_\varphi(x_0) \le 0, & x_0 \in \operatorname{Int} X, \\ \min\{F_\varphi(x_0), \Gamma_\varphi(x_0)\} \le 0, & x_0 \in \partial X. \end{cases}$$

**SUPERSOLUTION**

$w \in LSC(X)$; every $\varphi \in C^3(X)$ touching $w$ from below at $x_0$ must satisfy

$$\begin{cases} F_\varphi(x_0) \ge 0, & x_0 \in \operatorname{Int} X, \\ \max\{F_\varphi(x_0), \Gamma_\varphi(x_0)\} \ge 0, & x_0 \in \partial X. \end{cases}$$

**A continuous $w$ is a viscosity solution when it is both a subsolution and a supersolution.**

# A key result

**The value function is the unique viscosity solution of the degenerate HJB–Neumann problem.**

- The reflected controlled SDE is well posed.
- The value function is continuous and satisfies the DPP.
- A comparison principle yields uniqueness of the viscosity solution.

# Minimizing feedback controls

For $\psi \in C^3(X)$, define

$$\theta_\psi(x,u) := \langle \mu_X(x,u), D\psi(x)\rangle + L(x,u) + \tfrac{1}{2}\operatorname{Tr}\big(D^2\psi(x)[\sigma_X\sigma_X^\top](x,u)\big),$$

$$u^*_\psi(x) := \arg\min_{u\in U} \theta_\psi(x,u).$$

- For convex compact $U$ and $C^2$ dependence on $u$, uniform convexity in $u$ and uniform Lipschitz regularity of $D_u\theta_\psi$ in $x$ give a unique Lipschitz minimizer.
- With coefficients Lipschitz in $(x,u)$, the feedback $u_t = u^*_\psi(X_t)$ yields a unique strong reflected solution.

If $v^\beta \in C^3(X)$ satisfies HJB–Neumann classically, Itô's formula gives

$$u^*_t = u^*_{v^\beta}(X^{u^*}_t), \qquad J^\beta_x(u^*) = v^\beta(x).$$

In the viscosity proof, the regular minimizers are constructed for smooth test functions, since $v^\beta$ need not be differentiable.

# Periodic structure of the coefficients

Now, let us consider the multiscale problem. Here, $\epsilon > 0$ denotes the scaling parameter associated with the fast variable, with smaller values of $\epsilon$ corresponding to faster dynamics.

## 1 PERIODIC FAST STATE

$$Y_t^\varepsilon \in \mathbb{T}^{d_Y} = \mathbb{R}^{d_Y}/\mathbb{Z}^{d_Y}$$

so periodicity means $g(y+k) = g(y)$ for $k \in \mathbb{Z}^{d_Y}$.

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## 2 SLOW-COEFFICIENT DECOMPOSITION

$$\mu_X(x,y,u) = \mu_{SF}(x,y) + \mu_{SC}(x,u),$$
$$\sigma_X(x,y,u) = \sigma_{SF}(x,y) + \sigma_{SC}(x,u).$$

The diffusion components are non-correlated:

$$\sigma_{SC}(x,u)\sigma_{SF}^\top(x,y) = 0.$$

## 1 DOMAINS AND REGULARITY

$$\mu_Y : X \times \mathbb{T}^{d_Y} \to \mathbb{R}^{d_Y}, \qquad \sigma_Y : X \times \mathbb{T}^{d_Y} \to \mathbb{R}^{d_Y \times d_W}$$

Periodic in $y$, with $\mu_Y, \sigma_Y \in C^{1,2}(X \times \mathbb{T}^{d_Y})$.

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## 2 UNIFORM ELLIPTICITY

For some $0 < c_0 < 1$, uniformly in $(x, y) \in X \times \mathbb{T}^{d_Y}$,

$$c_0 \|\xi\|^2 \leq \langle \xi, [\sigma_Y \sigma_Y^\top](x, y)\xi \rangle \leq c_0^{-1} \|\xi\|^2, \qquad \xi \in \mathbb{R}^{d_Y}$$

# The slow–fast system

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \mu_X(X_t^\varepsilon, Y_t^\varepsilon, u_t)\,\mathrm{d}t + \sigma_X(X_t^\varepsilon, Y_t^\varepsilon, u_t)\,\mathrm{d}W_t - D\phi(X_t^\varepsilon)\,\mathrm{d}\ell_t^\varepsilon, \\ \mathrm{d}Y_t^\varepsilon = \varepsilon^{-1}\mu_Y(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t + \varepsilon^{-1/2}\sigma_Y(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}W_t. \end{cases}$$

- $X^\varepsilon$ evolves on the ordinary time scale and is reflected at $\partial X$.
- $Y^\varepsilon$ has drift of order $\varepsilon^{-1}$ and noise of order $\varepsilon^{-1/2}$.
- On the slow clock, the fast coordinate explores the torus repeatedly.

- Kushner ('90) developed a probabilistic approach to singularly perturbed stochastic control based on relaxed controls and weak convergence.

- Evans ('92) established key ideas for periodic homogenization. Alvarez and Bardi ('01, '03, '10) developed viscosity methods for singularly perturbed nonlinear HJB equations.

- Pavliotis and Stuart ('08) and Khasminskii ('11) provide the stochastic averaging framework that replaces fast dynamics by invariant averages. Bardi and Cesaroni ('09) and Bardi, Cesaroni and Manca ('10) apply this idea to multiscale control.
- Bardi and Kouhkouh ('23, '24) treat related recent degenerate settings.

**Our second paper adds a reflected state constraint on the slow variable and identifies the limit as an effective constrained control problem.**

# The product-space reduction

$$Z_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon) \in \widetilde{X} := X \times \mathbb{T}^{d_Y}, \quad \widetilde{\phi}(x,y) = \phi(x), \quad \partial\widetilde{X} = \partial X \times \mathbb{T}^{d_Y}.$$

- Fix $\varepsilon$ and combine the slow and fast coordinates into one state $Z^\varepsilon$.
- Reflection acts only on the slow coordinate because $\widetilde{\phi}(x,y) = \phi(x)$.
- Our previous key result now applies to the full system for this fixed $\varepsilon$.

# The singular HJB equation

$$\begin{cases} \beta v^{\beta,\varepsilon} - \mathcal{H}\Big(x, y, D_x v^{\beta,\varepsilon}, \varepsilon^{-1} D_y v^{\beta,\varepsilon}, D_x^2 v^{\beta,\varepsilon}, , \\ \qquad\qquad \varepsilon^{-1} D_y^2 v^{\beta,\varepsilon} \ \varepsilon^{-1/2} D_{xy}^2 v^{\beta,\varepsilon}\Big) = 0, \quad \text{on } X \times \mathbb{T}^{d_Y}, \\ \langle D_x v^{\beta,\varepsilon}, D\phi \rangle = h, \quad \text{on } \partial X \times \mathbb{T}^{d_Y}. \end{cases}$$

- Derivatives in the fast variable are amplified by $\varepsilon^{-1}$.
- Mixed derivatives appear at order $\varepsilon^{-1/2}$.
- For every $\varepsilon > 0$, our result for the single equation problem gives a unique viscosity solution.

# Inside the multiscale Hamiltonian

Here $g_x, g_y$ represent gradients, while $H_x, H_y, H_{xy}$ are Hessian blocks.

$$\mathcal{H}(x, y, g_x, g_y, H_x, H_y, H_{xy}) := \min_{u \in U} \Big\{ \mathcal{G}_X^u(x, g_x, H_x \mid y) + \mathcal{G}_Y(y, g_y, H_y \mid x) + \operatorname{Tr}\big(H_{xy}[\sigma_X \sigma_Y^\top](x, y, u)\big) + L(x, y, u) \Big\}.$$

- $\mathcal{G}_X^u$: slow generator, built from $\mu_X$ and $\sigma_X \sigma_X^\top$.
- $\mathcal{G}_Y$: fast generator, built from $\mu_Y$ and $\sigma_Y \sigma_Y^\top$.
- The mixed trace is the coupling created by the shared Brownian noise; $L$ is the running cost.
- The powers of $\varepsilon$ enter through the HJB arguments on the preceding slide.

# The singular limit ($\varepsilon \to 0$)

We show that there is a function $\overline{v}^{\beta}(x)$ (the effective value function) such that

$$v^{\beta,\varepsilon}(x,y) \xrightarrow[\varepsilon \downarrow 0]{\text{uniformly}} \overline{v}^{\beta}(x).$$

- The limiting value should no longer depend on the fast coordinate $y$.
- The influence on the fast dynamics must remain through its invariant distribution.
- Thus dimension reduction is an averaging problem plus a stability theorem.
- $\overline{v}^{\beta}(x)$ is the value function of a limiting control problem.

# The frozen fast process

$$\mathrm{d}Y_t^x = \mu_Y(x, Y_t^x)\,\mathrm{d}t + \sigma_Y(x, Y_t^x)\,\mathrm{d}W_t, \quad Y_t^x \in \mathbb{T}^{d_Y}.$$

- Freeze $x$: on the fast time scale, the slow state barely moves.
- Uniform ellipticity lets the fast process explore the torus.
- For each $x$, the frozen process has a unique invariant density $\rho_x$ on $\mathbb{T}^{d_Y}$.

# Invariant density and ergodic averages

$$\mathcal{L}_x^* \rho_x(y) = 0, \qquad \rho_x > 0, \qquad \int_{\mathbb{T}^{d_Y}} \rho_x(y)\,\mathrm{d}y = 1,$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}[\psi(Y_s^x)]\,\mathrm{d}s = \int_{\mathbb{T}^{d_Y}} \psi(y)\rho_x(y)\,\mathrm{d}y.$$

- The stationary Fokker–Planck equation determines $\rho_x$.
- Time averages of the fast motion become integrals over the torus.
- **we prove that $x \mapsto \rho_x$ is Lipschitz.**

# Regularity of the invariant density

For the effective problem to be well-posed and to obtain the necessary convergences, we need additional regularity assumptions on the coefficients of the fast dynamics:

$$\mu_Y, \sigma_Y \in C^{1,m+2}(X \times \mathbb{T}^{d_Y}), \qquad m > \frac{d_Y}{2} + 1,$$

Then we have:

- $x \mapsto \rho_x$, $x \mapsto D_y \rho_x$ and $x \mapsto D_y^2 \rho_x$ are continuous.
- The map $x \mapsto \rho_x$ is differentiable and
$$\sup_{(x,y) \in X \times \mathbb{T}^{d_Y}} \|D_x \rho_x(y)\| \leq C.$$

Hence
$$\sup_{y \in \mathbb{T}^{d_Y}} |\rho_{x_1}(y) - \rho_{x_2}(y)| \leq C \|x_1 - x_2\|.$$

# How the parameter estimate is obtained

1. **Elliptic regularity.** Uniform estimates control $\rho_x$ and its derivatives in the fast variable.
2. **Compactness.** Sobolev embedding and Arzelà–Ascoli give continuity with respect to $x$.
3. **Difference quotients.** For each coordinate $e_k$,

$$\rho_x^{h,k}(y) := \frac{\rho_{x+he_k}(y) - \rho_x(y)}{h}$$

   solves a linear elliptic equation on the torus and has zero mean.
4. **Uniqueness and estimates.** The Fredholm alternative and the weak maximum principle identify the limit and bound $\partial_{x_k}\rho_x$ uniformly.

# Averaged drift and running cost

$$\overline{\mu}_X(x) = \int_{\mathbb{T}^{d_Y}} \mu_{SF}(x,y)\rho_x(y)\,\mathrm{d}y,$$

$$\overline{L}(x) = \int_{\mathbb{T}^{d_Y}} L_{SF}(x,y)\rho_x(y)\,\mathrm{d}y.$$

- Freeze the slow state $x$.
- Average the fast-state contribution against $\rho_x$.
- The control-dependent slow terms remain unchanged.

# Averaging the covariance

$$\bar{a}_X(x) := \int_{\mathbb{T}^{d_Y}} [\sigma_{SF}\sigma_{SF}^\top](x,y)\rho_x(y)\,\mathrm{d}y,$$

$$\bar{\sigma}_X(x)\bar{\sigma}_X(x)^\top = \bar{a}_X(x).$$

**Average $\sigma\sigma^\top$ – not $\sigma$ itself.**

- After averaging the covariance, choose the Lipschitz square root $\bar{\sigma}_X$.
- This preserves the correct quadratic variation of the slow motion.

# From invariant density to effective coefficients

$$\overline{\mu}_X(x) = \int \mu_{SF}(x,y)\rho_x(y)\,\mathrm{d}y,$$

$$\overline{L}(x) = \int L_{SF}(x,y)\rho_x(y)\,\mathrm{d}y,$$

$$\overline{a}_X(x) = \int [\sigma_{SF}\sigma_{SF}^\top](x,y)\rho_x(y)\,\mathrm{d}y,$$

- The Lipschitz estimate for $\rho_x$ implies that $\overline{\mu}_X$, $\overline{L}$ and $\overline{a}_X$ are Lipschitz in $x$.
- The effective SDE needs a Lipschitz factor $\overline{\sigma}_X\overline{\sigma}_X^\top = \overline{a}_X$.
- A Lipschitz covariance need not have a Lipschitz square root near degeneracy. We record this as an additional requirement.
- If $\overline{a}_X$ is uniformly positive definite, its symmetric square root provides the required $\overline{\sigma}_X$.

# The effective reflected control problem

$$
\begin{aligned}
\mathrm{d}\overline{X}_t^u = & \left[\overline{\mu}_X(\overline{X}_t^u) + \mu_{SC}(\overline{X}_t^u, u_t)\right] \mathrm{d}t \\
& + \left[\overline{\sigma}_X(\overline{X}_t^u) + \sigma_{SC}(\overline{X}_t^u, u_t)\right] \mathrm{d}W_t \\
& - D\phi(\overline{X}_t^u)\, \mathrm{d}\overline{\ell}_t^u.
\end{aligned}
$$

- The fast state is gone.
- Control, reflection and the boundary regulator are preserved.
- Only the fast-dependent coefficients have been replaced by averages.

# Well-posedness and HJB characterization

1. **Effective dynamics.** The Lipschitz coefficients give a unique strong solution of the reflected effective SDE.

2. **Continuous value function.** The averaged running cost is Lipschitz, so the effective costs are equicontinuous in the initial state and $\overline{v}^{\beta} \in C(X)$.

3. **Dynamic programming.** The effective value function satisfies the same DPP as the one-scale problem.

4. **Bellman structure.**

$$\sigma_{SC}(x,u)\overline{\sigma}_X(x)^{\top} = 0,$$

and the averaged Hamiltonian is the Hamiltonian of the effective control problem. Hence $\overline{v}^{\beta}$ is the unique viscosity solution of the effective HJB–Neumann equation.

# The cell-$t$ problem

Fix $p = (x, g, H)$ and set

$$\mathfrak{H}_p(y) := \min_{u \in U} \{\mathcal{G}_X^u(x, g, H \mid y) + L(x, y, u)\}.$$

We obtain the effective halmitonian from ergodicity as

$$\overline{\mathcal{H}}(x, g, H) = \lim_{t \to \infty} \frac{w_p(t, y)}{t} = \int_{\mathbb{T}^{d_Y}} \mathfrak{H}_p(y) \rho_x(y) \, \mathrm{d}y,$$

where

$$w_p(t, y) = \int_0^t \mathbb{E}[\mathfrak{H}_p(Y_s^{x,y})] \, \mathrm{d}s.$$

It solves

$$\begin{cases} -\partial_t w_p + \mathcal{L}_x w_p + \mathfrak{H}_p(y) = 0, & (t, y) \in \mathbb{R}_+ \times \mathbb{T}^{d_Y}, \\ w_p(0, y) = 0, & y \in \mathbb{T}^{d_Y}. \end{cases}$$

Here $\mathcal{L}_x$ is the generator of the frozen fast process, and $Y^{x,y}$ denotes that process started from $y$.

# The effective Hamiltonian

$$\mathfrak{H}_p(y) := \min_{u \in U} \left\{ \mathcal{G}_X^u(\overline{x}, g, H \mid y) + L(\overline{x}, y, u) \right\},$$

$$\overline{\mathcal{H}}(\overline{x}, g, H) = \int_{\mathbb{T}^{d_Y}} \mathfrak{H}_p(y) \rho_{\overline{x}}(y) \, \mathrm{d}y.$$

- Freeze the slow variable $x$, gradient $g$ and Hessian $H$.
- Minimize over the control at each fast state $y$.
- Average the resulting quantity against $\rho_x$.

# Main convergence theorem

**The multiscale value functions converge uniformly to the value function of the effective reflected control problem.**

$$\lim_{\varepsilon \downarrow 0} \sup_{(x,y) \in X \times \mathbb{T}^{d_Y}} \left| v^{\beta,\varepsilon}(x,y) - \overline{v}^{\beta}(x) \right| = 0.$$

- The convergence is uniform in both the slow state $x$ and the fast state $y$.
- The limit is a genuine stochastic-control value, not only a formal PDE limit.

# Proof strategy

1. **Relaxed half-limits**

   Construct upper and lower candidates without assuming convergence.

2. **Perturbed test functions**

   Insert a fast corrector obtained from the cell problem.

3. **Comparison**

   Show that every subsolution lies below every supersolution.

# Equiboundedness and relaxed limits

The bounded running and boundary costs imply, uniformly for $0 < \varepsilon \leq 1$,

$$\sup_{(x,y)\in X\times\mathbb{T}^{d_Y}} |v^{\beta,\varepsilon}(x,y)| \leq \frac{C_0}{\beta} + C_1.$$

Without assuming convergence, define

$$v^{\beta}_{\inf}(x_0) := \liminf_{\substack{\varepsilon\to 0\\ x\to x_0}} \inf_{y\in\mathbb{T}^{d_Y}} v^{\beta,\varepsilon}(x,y),$$

$$v^{\beta}_{\sup}(x_0) := \limsup_{\substack{\varepsilon\to 0\\ x\to x_0}} \sup_{y\in\mathbb{T}^{d_Y}} v^{\beta,\varepsilon}(x,y).$$

The goal is to prove that $v^{\beta}_{\sup}$ is a viscosity subsolution and $v^{\beta}_{\inf}$ is a viscosity supersolution of the effective HJB.

# Perturbed test functions at the boundary

Let $\varphi$ touch $v^{\beta}_{\sup}$ from above at $x_0 \in \partial X$. Set $p = (x_0, D\varphi(x_0), D^2\varphi(x_0))$.

- Testing with $\varphi(x)$ alone removes the fast derivatives but leaves the oscillating term $\mathfrak{H}_p(y)$, rather than its invariant average.
- For fixed $t_0$, introduce a perturbation that vanishes as $\varepsilon \to 0$:

$$\varphi^{\varepsilon}_{p,t_0}(x,y) := \varphi(x) + \frac{\varepsilon}{t_0}\int_0^{t_0} w_p(s,y)\,\mathrm{d}s.$$

  The $\varepsilon^{-1}$ factor amplifies this small correction. The cell-$t$ equation gives the cancellation

$$\mathfrak{H}_p(y) + \underbrace{\varepsilon^{-1}\mathcal{L}_{x_0}\varphi^{\varepsilon}_{p,t_0}}_{\text{fast correction}} = \frac{w_p(t_0,y)}{t_0} \xrightarrow[t_0\to\infty]{\text{uniformly in } y} \overline{\mathcal{H}}(p).$$

- The corrector depends only on $y$, so $D_x\varphi^{\varepsilon}_{p,t_0} = D\varphi$ and $D^2_{xy}\varphi^{\varepsilon}_{p,t_0} = 0$. The Neumann boundary operator is therefore unchanged.

First choose $t_0$ large; then let $\varepsilon \to 0$.

# Example

$$X = \overline{B}(0, \alpha) \subset \mathbb{R}^2, \qquad Y \in \mathbb{T}^2, \qquad U = [u_a, u_b],$$

$$\phi(x) = \frac{1}{2\alpha} e^{\alpha^2 - \|x\|^2} (\|x\|^2 - \alpha^2),$$

$$L(x, y, u) = (\theta_d - u)^2, \qquad h(x) = \theta_e x_1^2 x_2^2.$$

- The slow state is reflected inside a disk in $\mathbb{R}^2$.
- The fast state evolves on the two-dimensional torus.
- The running cost favors a target control level; the boundary cost depends on position.

# Dynamics in the example

$$\mu_X(x,y,u) = \theta_a \begin{bmatrix} x_1 \sin(2\pi y_1)\cos(2\pi y_2) \\ x_2 \cos(2\pi y_1)\sin(2\pi y_2) \end{bmatrix} + \theta_b u \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\sigma_X(x,y,u) = \sigma_X \begin{bmatrix} \sqrt{u}\, x_1 & 0 \\ 0 & (\sin(2\pi y_1)\sin(2\pi y_2) + 2)x_2 \end{bmatrix}.$$

$$\mu_Y(x,y) = \theta_c \|x\|^2 \cos(2\pi(y_1 - y_2)) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \sigma_Y(x,y) = \sigma_Y I_2.$$

- The control changes both the slow drift and a degenerate diffusion.
- Uniformly elliptic fast noise makes the torus dynamics ergodic.

# The invariant averages $\kappa_1, \kappa_2, \kappa_3$

$$\kappa_1(x) = \int_{\mathbb{T}^2} \sin(2\pi y_1)\cos(2\pi y_2)\rho_x(y)\,\mathrm{d}y,$$

$$\kappa_2(x) = \int_{\mathbb{T}^2} \cos(2\pi y_1)\sin(2\pi y_2)\rho_x(y)\,\mathrm{d}y,$$

$$\kappa_3(x) = \left[\int_{\mathbb{T}^2} \big(\sin(2\pi y_1)\sin(2\pi y_2) + 2\big)^2 \rho_x(y)\,\mathrm{d}y\right]^{1/2}.$$

- $\kappa_1$ and $\kappa_2$ are invariant means of the periodic drift modes.
- $\kappa_3$ is a root-mean-square diffusion amplitude.

# Effective coefficients in the example

$$\overline{\mu}_X(x) = \theta_a \begin{bmatrix} x_1\kappa_1(x) \\ x_2\kappa_2(x) \end{bmatrix},$$

$$\overline{\sigma}_X(x) = \sigma_X \begin{bmatrix} 0 & 0 \\ 0 & \kappa_3(x)x_2 \end{bmatrix}.$$

- All dependence on the eliminated fast state is carried by $\rho_x$ through $\kappa_1, \kappa_2, \kappa_3$.
- The control-dependent part of the coefficients is unchanged.
- The effective model remains degenerate and reflected.

# The effective Hamiltonian

$$\overline{\mathcal{H}}(x, g, H) = \min_{u \in U} \left\{ \left\langle \theta_a \begin{bmatrix} x_1 \kappa_1(x) \\ x_2 \kappa_2(x) \end{bmatrix} + \theta_b u \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, g \right\rangle + (u - \theta_d)^2 + \frac{\sigma_X^2}{2} \operatorname{Tr}\left( H \begin{bmatrix} u x_1^2 & 0 \\ 0 & \kappa_3^2(x) x_2^2 \end{bmatrix} \right) \right\}.$$

- Averaged periodic drift enters through $\kappa_1$ and $\kappa_2$.
- The controlled variance remains proportional to $u x_1^2$.
- The second variance component contains $\kappa_3^2(x) x_2^2$.

# Effective HJB and convergence

$$\beta \overline{v}^{\beta} - \overline{\mathcal{H}}(x, D\overline{v}^{\beta}, D^2\overline{v}^{\beta}) = 0, \qquad \langle D\overline{v}^{\beta}, D\phi \rangle = \theta_e x_1^2 x_2^2 \quad \text{on } \partial X.$$

$$\lim_{\varepsilon \downarrow 0} \sup_{(x,y) \in X \times \mathbb{T}^{d_Y}} \left| v^{\beta,\varepsilon}(x,y) - \overline{v}^{\beta}(x) \right| = 0.$$

**The fast variable disappears from the state, but survives through $\rho_x$ and $\kappa_1, \kappa_2, \kappa_3$.**

# The example is fully explicit

$$\operatorname{div}_y \mu_Y(x,y) = \theta_c \|x\|^2 (\partial_{y_1} + \partial_{y_2}) \cos\big(2\pi(y_1 - y_2)\big) = 0,$$

$$\rho_x \equiv 1 \quad \Longrightarrow \quad \kappa_1 = \kappa_2 = 0, \qquad \kappa_3 = \frac{\sqrt{17}}{2}.$$

## HOMOGENIZED VALUE EQUATION

$$\beta \overline{v}^\beta(x) - \min_{u \in U} \left\{ \theta_b u \big( x_1 \partial_1 \overline{v}^\beta + x_2 \partial_2 \overline{v}^\beta \big) + (u - \theta_d)^2 \right.$$
$$\left. + \frac{\sigma_X^2}{2} \left( u x_1^2 \partial_{11} \overline{v}^\beta + \frac{17}{4} x_2^2 \partial_{22} \overline{v}^\beta \right) \right\} = 0, \qquad x \in \operatorname{Int}(X),$$

$$\big\langle D\overline{v}^\beta(x), D\phi(x) \big\rangle = \theta_e x_1^2 x_2^2, \qquad x \in \partial X.$$

# Numerical illustration: setup

We fix an illustrative parameter choice for the homogenized problem:

$$\alpha = \beta = \theta_b = 1, \qquad \sigma_X = 0.4, \qquad \theta_d = 0.7, \qquad \theta_e = 4.$$

$$X = \overline{B}(0,1), \qquad U = [0,1], \qquad \overline{v} := \overline{v}^{\,1}.$$

To include $u = 0$, we extend the paper's assumption $u_a > 0$.

$$J_x(u) = \mathbb{E}_x\left[\int_0^\infty e^{-t}(u_t - 0.7)^2\,\mathrm{d}t + \int_0^\infty e^{-t} 4\big(X_1(t)X_2(t)\big)^2\,\mathrm{d}\ell_t\right],$$

$$\overline{v}(x) = \inf_{u\in\mathcal{U}} J_x(u).$$

## DETERMINISTIC BENCHMARKS

$$u_t \equiv 0, \qquad u_t \equiv 1, \qquad u_t \equiv 0.7, \qquad u_r(t) = \min\{t/2, 1\}.$$

For $u = 0.7$, the running cost vanishes; the boundary cost remains.

# Numerical comparison

| $x$ | $\overline{v}(x)$ | $J_x(0)$ | $J_x(1)$ | $J_x(0.7)$ | $J_x(u_r)$ |
|---|---|---|---|---|---|
| $(0,0)$ | **0.000** | 0.490 | 0.090 | 0.000 | 0.182 |
| $(0.25,0.25)$ | **0.064** | 0.492 | 0.264 | 0.084 | 0.234 |
| $(0.5,0.5)$ | **0.226** | 0.539 | 0.527 | 0.288 | 0.341 |
| $(0.7,0.7)$ | **0.448** | 0.705 | 0.788 | 0.532 | 0.531 |
| $(0.8,0.4)$ | **0.277** | 0.570 | 0.545 | 0.332 | 0.382 |
| $(0.4,0.8)$ | **0.240** | 0.581 | 0.501 | 0.284 | 0.361 |

# Feedback control in the example

If $\overline{v}$ is sufficiently smooth, pointwise minimization of the Hamiltonian gives the candidate feedback

$$u^*(x) = \min\{1, \max\{0,\ 0.7 - \tfrac{1}{2}\, x \cdot D\overline{v}(x) - 0.04 x_1^2 \partial_{11}\overline{v}(x)\}\}\,.$$

- Policy iteration provides a minimizing numerical policy $\widehat{u}(x)$ for the discrete problem, applied in feedback form as $u_t = \widehat{u}(X_t)$.
- At $x = (0.5, 0.5)$, compare the simulated feedback cost with the HJB value estimate:

$$J_x(\widehat{u}) \approx 0.2267, \qquad \overline{v}(x) \approx 0.2263.$$

The 95% statistical half-width for the simulated cost is 0.0006.